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ORBITAL MOMENTUM OF VORTICES AND TEXTURES DUE TO SPECTRAL FLOW THROUGH THE GAP NODES: EXAMPLE OF THE ^{3}He-A CONTINUOUS VORTEX

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The orbital momentum of the axisymmetric textures and vortices in fermi superfluids and superconductors is discussed on the example of ^{3}He-A. If there are no zeroes in the quasiparticle spectrum, the orbital momentum of the texture is robust, ie it is not sensitive to the change of the texture, provided that the axial symmetry is not violated. If zeroes exist or there is an anomalous branch of the low-energy fermions in the vortex core, the orbital momentum depends on texture. This dependence comes from accumulation of the fermionic topological charge induced by the texture. The change of the orbital momentum in texture occurs by spectral flow through the nodes or along the anomalous branch.

1. Introduction. Axial symmetry charge

Here we consider the static orbital angular momentum of the axisymmetric distribution of the order parameter in Fermi superfluids and superconductors. We apply the general approach of the spectral flow, which is valid both for quantized vortices in conventional superconductors and ^{3}He-B, where vortices have singular cores, and for the continuous order parameter texture, which can exist in ^{3}He-A. The obtained general result, when applied to the ^{3}He-A texture, allows us to treat the angular momentum paradox in ^{3}He-A.

The paradox of the orbital momentum in Fermi pair-correlated states has a long hystory, started in 1961 when Anderson and Morel[1] introduced the anisotropic state further realized as the A-phase of superfluid ^{3}He. Each Cooper pair in this state has an orbital momentum $\hbar\mathbf{l}$, where $\mathbf{l}$ is the unit vector of the orbital anisotropy. Estimations of the total angular momentum of the system varies from $(N/2)\hbar\mathbf{l}$, which corresponds to the momentum $\hbar/2$ per each of N particles of the system (see [2]), to much smaller quantity $\sim N(\Delta_0/E_F)^2\hbar\mathbf{l}$ (see recent paper [3], here Δ_0 is the gap amplitude, which is much smaller than the Fermi energy E_F). The latter estimation corresponds to the space integral of the intrinsic dynamical momentum, found by Cross [4], which is related to the inertia of the $\mathbf{l}$ precession.

We show here that this paradox is closely related to the axial anomaly, which appears either due to zeroes in the quasiparticle spectrum [5–7] or due to quantized vortices[8]. In both cases the spectral flow through the gap nodes or along the anomalous branch of fermions within the vortex core leads to accumulation of the orbital momentum. The latter plays the part of the topological charge induced in the Fermi sea by the texture of the order parameter (on the fermionic topological charge formalism in superfluid ^{3}He see [6]).

The relevant fermionic charge in the axisymmetric system is related to the residual symmetry of the system. This is the generalized angular momentum, expressed in terms of the angular momentum and the particle number operators (see eg review [9]):

$$Q = L_z - (n/2)N \quad . \tag{1.1}$$

Here n is an integer: $n = 1$ for the homogeneous A-phase with $\mathbf{l} = \hat{z}$, where the rotational symmetry $SO(3)_L$ is spontaneously broken together with the gauge symmetry $U(1)_N$, but the combined symmetry with the generator $Q = L_z - (1/2)N$ is conserved. This means that the action of Q on the (multicomponent) order parameter Ψ annihilates the order parameter: $Q\Psi = 0$, ie the A-phase order parameter does not change if the rotation is accompanied by the proper gauge transformation, which is generated by N and leads to the change of the phase of the order parameter.

The Eq.(1.1) can be also applied to the inhomogeneous vacuum [9], eg to quantized vortices in conventional s-wave pair-correlated state: in this case n is the winding number of the vortex. In the inhomoheneous case the generator L_z contains two terms: $L_z = L_z^{internal} + L_z^{external}$, where $L^{internal}$ is the generator of the orbital rotations in the order parameter space (in the isotopic $SO(3)_L$ group), while $L^{external} = -i\mathbf{r} \times \partial_{\mathbf{r}}$ is the generator of the coordinate r rotations. The axisymmetric, or Q-symmetric state means that $Q\Psi(\mathbf{r}) = 0$.

This symmetry of the vacuum tells that Q is the conserved and integer (or half of odd integer) quantum number, but this does not mean that the Q-charge of the vacuum should be exactly zero. In the pure fermionic description (Section 4) the total charge of the vacuum is

$$< vac|Q|vac >= \sum_{Q,p_z,s} Q\theta(-E_{Q,p_z,s}) \tag{1.2}$$

where $E_{Q,p_z,s}$ are the energy eigenvalues for fermions in the axisymmetric field of the order parameter (in addition to the quantum number Q, there are the other quantum numbers: the linear momentum p_z along the symmetry axis z, the radial quantum number s, etc.); $\theta(-E_{Q,p_z,s})$ is the step function of the energy, which shows that only the negative energy states contributes to the vacuum charge. The charge of the vacuum can be nonzero if some descrete symmetry is broken and $E_{Q,p_z,s} \neq E_{-Q,\pm p_z,s}$.

One can find the condition when this charge is zero, which for the A-phase state means that the total angular momentum $L_z =< vac|L_z|vac >= (n/2)N$ in accordance with Ref.[2]. This condition is related to the adiabatic process, which means that during the process there is no level flow from or into the vacuum state and thus the fermionic charge is conserved in this process. This takes place for example if there is a gap in the fermionic spectrum. For the A-phase state this occurs in the limit case of the Bose condensate of the isolated Cooper molecules each with momentum $\hbar$ (or in a thin film where the gap nodes disappear due to transverse quantization, see Chapter 9 of Ref.[7]). Let us start from the Bose condensate as an initial state, which has $Q = 0$ (and thus $L_z = N/2$), and transform this state adiabatically into the real ^{3}He-A without violation of the axisymmetry: then in the final ^{3}He-A state one also has $Q = 0$. The problem is, however, that during this continuous transformation the gap nodes appear at some moment and the process can loose the adiabadicity, since the spectral flow through the nodes can emerge in the bulk liquid or at the boundary of the system.

We consider how the spectral flow leads to nonzero Q in the vacuum state of the axisymmetric texture in the arbitrary pair-correlated system and apply the result to the continuous vortex in the A-phase. This is the simplest quantized vortex, in which the microscopic calculations (Section 4) can be completed and compared to the phenomenological hydrodynamical approach (Section 3). But let us first recall how the spectral flow modifies the linear momentum in the A-phase.

2. Linear momentum anomaly in ^{3}He-A

Let us start from the Bose condensate of the isolated Cooper molecules with the symmetry of the A-phase or from the A-phase state with negative chemical potential $\mu < 0$, which fermionic spectrum $E = \sqrt{(p^2/2m_3 - \mu)^2 + c^2(\mathbf{l} \times \mathbf{p})^2}$ also has no nodes. One can adiabatically transform these two states into each other and therefore they have identical properties. The mass current (or the density of the linear momentum) in these node-free superfluids at $T = 0$ is

$$\mathbf{j}_{node-free} = \frac{\hbar}{2m_3}\rho \mathbf{v}_s + \frac{1}{2}\vec{\nabla} \times \mathbf{L}_{node-free} \quad . \tag{2.1}$$

The first term is dictated by the Galilean invariance, here $\mathbf{v}_s$ is the the superfluid velocity in units of $\hbar/2m_3$. The vector $\mathbf{L}$ is the density of the angular momentum, which for the nodes-free states is

$$\mathbf{L}_{node-free} = \frac{\hbar}{2m_3}\rho \ \mathbf{l} \quad . \tag{2.2}$$

Let us now continuously transform the node-free liquid into the real A-phase by changing the chemical potential from negative to positive. At $\mu > 0$ the gap nodes appear at two points $\mathbf{p} = \pm p_F \mathbf{l}$ where $p_F^2/2m_3 = \mu$. Near each node the fermions can be described as chiral Weyl fermions moving in the "electromagnetic" field $\mathbf{A} = p_F \mathbf{l}$ produced by the $\mathbf{l}$ texture[7]. If $(\partial_t \mathbf{A} \cdot (\vec{\nabla} \times \mathbf{A})) \neq 0$ there is an effect of axial anomaly [10]: the spectral flow of fermions leads to creation of quasiparticles from the vacuum. In ^{3}He-A each created quasiparticle carries the linear momentum $p_F \mathbf{l}$. This results in the production of the net quasiparticle linear momentum:

$$\partial_t \mathbf{P}_{qp} = \frac{1}{2\pi^2}\int d^3r \ \ p_F \mathbf{l} \ \ (\partial_t \mathbf{A} \cdot (\vec{\nabla} \times \mathbf{A})) \quad . \tag{2.3}$$

Since the total linear momentum is conserved, this means the transfer of the momentum from the collective variables of the order parameter to the system of quasiparticles.

Let us take the arbitrary but fixed $\mathbf{l}(\mathbf{r})$-texture in the node-free state and consider the transformation into the real A-phase in such a way that only μ changes with time. At some moment $t = t_0$ the Fermi momentum appears which then changes from $p_F = 0$ at $t = t_0$ to its equilibrium value $p_F(\infty)$ in the real A-phase at $t = \infty$. In this process $\partial_t \mathbf{A} = \mathbf{l}\partial_t p_F$ and one has the following change of the total momentum of the texture as compared to Eq.(2.1)

$$\mathbf{P}(\infty) - \mathbf{P}(t_0) = -\int_{t_0}^{\infty} dt \ \ \partial_t \mathbf{P}_{qp} = -\frac{1}{2\pi^2}\int_{t_0}^{\infty} dt \int d^3r \ \ p_F^2 \partial_t p_F \ \ \mathbf{l} \ \ (\mathbf{l} \cdot (\vec{\nabla} \times \mathbf{l})) =$$

$$= -\frac{1}{2}\int d^3r \ \ C_0 \ \mathbf{l} \ \ (\mathbf{l} \cdot (\vec{\nabla} \times \mathbf{l})) \quad , \quad C_0 = \frac{1}{3\pi^2}p_F^3(\infty) \quad , \tag{2.4}$$

where $P(t_0) = \int d^3r\ j_{node-free}$ is the anomaly-free momentum in Eq.(2.1). The extra mass current in the A-phase

$$j_{anomalous} = -\frac{1}{2} C_0 \mathbf{l} \left(\mathbf{l} \cdot (\vec{\nabla} \times \mathbf{l}) \right) \quad . \tag{2.5}$$

results from the helicity of the A field (on the role of the helicity in particle physics see ref.[11]).

3. Angular momentum of the ^{3}He-A texture: phenomenological approach

Let us estimate the Q-charge of the vacuum in the A-phase for different continuous axisymmetric l-textures, which can be obtained by continuous deformation of the homogeneous vacuum with $\mathbf{l} = \hat{z}$.

The general solution of the axisymmetry equation $Q\mathbf{l}(\mathbf{r}) = 0$ for the l texture is

$$\mathbf{l} = \hat{z} \cos \eta(r) + \sin \eta(r) (\hat{r} \cos \alpha(r) + \hat{\phi} \sin \alpha(r)) \quad . \tag{3.1}$$

We take $\eta(0) = 0$ to have $\mathbf{l}(r = 0) = \hat{z}$ in the center of the vessel, this is required by the continuity of the deformation of the homogeneous state with $\mathbf{l} = \hat{z}$.

If $\eta(r = r_0) = \pi$ the texture represents the continuous Anderson–Toulouse–Chechetkin 4π vortex in ^{3}He-A [12] with

$$n = \frac{1}{2\pi} \oint_{r > r_0} d\mathbf{r} \cdot \mathbf{v}_s = \frac{1}{2\pi} \int d\mathbf{S} \cdot \vec{\nabla} \times \mathbf{v}_s = \frac{1}{2\pi} \int dx\ dy\ \mathbf{l} \cdot (\partial_x \mathbf{l} \times \partial_y \mathbf{l}) = 2 \quad , \tag{3.2}$$

and with the core radius r_0. Here we used the Mermin–Ho relation [13]

$$\vec{\nabla} \times \mathbf{v}_s = \frac{1}{2} e_{ijk} l_i \vec{\nabla} l_j \times \vec{\nabla} l_k \quad . \tag{3.3}$$

and the expression for the topological invariant which describes the mapping $S^2 \to S^2$ of the vortex cross-section to the sphere S^2 of the unit vector $\mathbf{l} \cdot \mathbf{l} = 1$. The invariant shows that within the continuous 4π-vortex the whole area 4π of the sphere is swept once. For simplicity further we consider the coordinate independent α.

In the phenomenological description the orbital momentum is given by the momentum of the current: $\mathbf{L} = \int d^3r\ \mathbf{r} \times \mathbf{j}$. The integration of the regular terms, Eq.(2.1), after integration by part and using the boundary condition $\rho(R) = 0$ (there is no particles outside the vessel of radius R), gives the standard contribution to the angular momentum: $L_z(\text{regular}) = \frac{1}{2}N$. Thus the charge Q of the axisimmetric vacuum is given by the orbital momentum of the anomalous current:

$$< vac|Q|vac > = \int d^3r\ \hat{z} \cdot (\mathbf{r} \times j_{anomalous}) = -\frac{1}{2} \int d^3r\ C_0\ (\hat{z} \cdot (\mathbf{r} \times \mathbf{l}))\ (\mathbf{l} \cdot (\vec{\nabla} \times \mathbf{l})) =$$

$$= -\pi L \int_0^{r_0} dr\ r^2 C_0 \sin^2 \alpha \sin \eta\ (\partial_r \eta + \frac{\sin \eta\ \cos \eta}{r}) \quad , \tag{3.4}$$

(here L is the length of the vortex). This means that if the l texture contains a helix (ie if $\sin \alpha \neq 0$) the total momentum of the vortex texture in the A-phase is reduced as compared with $(1/2)N$ calculated for the node-free models.

In the following Section this phenomenological expression is rederived from the general expression obtained using the spectral flow along the anomalous branch of the fermions localized in the vortex core.

4. Orbital momentum from fermion zero modes on vortices

In particle physics the fermion zero modes on strings are the p_z-modes, ie they correspond to branches of spectrum of fermions localized in strings, $E_{Q,p_z,s}$, which cross zero energy as a function of continuous parameter p_z. For the condensed matter strings, vortices in the pair-correlated systems, the important zero modes are Q-modes [14], the branches of spectrum $E_{Q,p_z,s=0}$, which cross zero energy as a function of parameter Q. In most cases the charge Q can be considered as continuous. The Q zero modes in condensed matter have the property of the p_z modes in particle physics: the algebraic sum of zero modes is nonzero and is defined by the winding number n of the vortex [8]. This means that the number ν of the negative fermionic levels with given p_z is different for large positive and large negative Q:

$$\nu(p_z, Q = +\infty) - \nu(p_z, Q = -\infty) = 2n \quad , \tag{4.1}$$

and the branch $E_{Q,p_z,s=0}$ crosses zero at some $Q = Q_0(p_z)$.

In the most symmetric vortices $Q_0(p_z) = 0$, ie the energy spectrum of the fermions on the anomalous branch, $E_{Q,p_z,s=0} = Q\omega(p_z)$, crosses zero at $Q = 0$. However, if some discrete symmetry is broken in the vortex core, then $Q_0(p_z) \neq 0$. Such situation was found in the continuous vortices in the A-phase if the helicity of vector $\mathbf{l}$ is nonzero [15]. According to Kopnin [15, 16]

$$Q_0(p_z) = r(p_z) \ \sin\alpha \ \sqrt{p_F^2 - p_z^2} \quad , \tag{4.2}$$

where $r(p_z)$ is the radius at which

$$\cos \ \eta(r) = \frac{p_z}{p_F} \quad , \tag{4.3}$$

and lowest energy levels with the radial quantum number $s = 0$ are given by

$$E(Q, p_z, s = 0) = \frac{\Delta_0}{p_F \ r(p_z) \ \cos\alpha} \ (Q - Q_0(p_z)) \quad . \tag{4.4}$$

Though Q is discrete, the distance between the Q levels $\Delta_0/(p_F \ r(p_z) \ \cos\alpha)$ is very small compared with the gap amplitude Δ_0, which means that the effective Q is large and can be considered as continuous.

Since $Q_0(p_z)$ depends on $\sin\alpha$, during the evolution of the vortex structure, Q levels cross the zero energy and this leads to the accumulation of the charge Q in the vacuum, which was phenomelogically discussed in Sec.3. In terms of the fermionic levels the rate of the charge Q production

$$\partial_t < vac|\mathrm{Q}|vac >= \sum_{Q,p_z} Q\partial_t\nu(Q, p_z) \quad , \tag{4.5}$$

can be found from the following consideration. If one changes $Q_0(p_z)$ due to the modification of the vortex, eg due to the change of α, the rate of the flow of the Q levels through zero is $\partial_t Q_0(p_z)$. Since at each event the charge $Q_0(p_z)$ is transferred from the vacuum to the fermionic degrees of freedom, the total rate of the charge transfer is

$$\partial_t < vac|\mathrm{Q}|vac >= \sum_{p_z} Q_0(p_z)\partial_t Q_0(p_z) \quad . \tag{4.6}$$

Thus if one starts from the most symmetric vortex and continuously transfers this state into the vortex with broken symmetry, one obtains the following general result for the charge Q of the vortex:

$$< vac|Q|vac >= \frac{1}{2}\sum_{p_z} Q_0^2(p_z) \quad . \tag{4.7}$$

Now we can apply this general result to the A-phase vortex. Using Eq.(4.2) one obtains the Q charge of the helical texture:

$$< \mathrm{vac}|Q|\mathrm{vac} >= \sin^2\alpha \ L \int \frac{dp_z}{2\pi} \ r^2(p_z) \ (p_F^2 - p_z^2) \quad . \tag{4.8}$$

One can show that this is just the Eq.(3.4). According to equation (4.2), the function $r(p_z)$ is the inverse function of $p_z(r) = p_F(r)\cos\eta(r)$. Adding to Eq.(4.7) the factor $1 = \int_0^R dr \ \delta(r - r(p_z))$ (where R is the radius of the vessel), one obtains

$$\int \frac{dp_z}{2\pi} \ r^2(p_z) \ (p_F^2 - p_z^2) = \int \frac{dp_z}{2\pi} \int_0^R dr \ \delta(r - r(p_z)) \ r^2(p_z) \ (p_F^2 - p_z^2) =$$

$$= \frac{1}{2\pi} \int_0^R dr \ r^2 p_F^2(r) \ \sin^2\eta \ \partial_r(p_F\cos\eta) \ =$$

$$= \frac{1}{2\pi} \int_0^R dr \ [\ r^2 p_F^3(r) \ \sin^2\eta \ \partial_r\cos\eta + r^2\sin^2\eta \ \cos\eta \ \partial_r \frac{p_F^3}{3} \ ] \ . \tag{4.9}$$

The second term is integrated by parts using the condition $p_F = 0$ outside the vessel and one has

$$\int \frac{dp_z}{2\pi} \ r^2(p_z) \ (p_F^2 - p_z^2) = -\frac{1}{3\pi} \int_0^{r_0} dr \ r^2 p_F^3 \ \sin\eta \ (\partial_r\eta + \frac{\sin\eta \ \cos\eta}{r}) \quad , \tag{4.10}$$

which corresponds to Eq.(3.4).

5. Conclusion

The orbital momentum of the axisymmetric vacuum in the pair-correlated fermionic system is $L_z = (n/2)N + Q$, where N is the number of particles, n is integer and Q is the conserved fermionic charge in the axisymmetric vacuum. In the presence of the gap nodes the charge Q is given by Eq.(4.7) and depends on the texture. The gap nodes, which give rise to $Q \neq 0$, can exist (i) in the bulk liquid (like in the A-phase); (ii) within the core of vortices; (iii) on the surface of container (the effect of the surface will be discussed later). When the texture changes, the charge Q is accumulated by the flow of Q levels through zeroes. This occurs only if some discrete symmetry is violated in the texture; in ^{3}He-A the effect exists only in the presence of the helical texture, ie with $\mathbf{l}\cdot(\vec{\nabla}\times\mathbf{l}) \neq 0$.

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О ПРИМЕНИМОСТИ "SLAVE-BOSON"-МЕТОДА К РАСЧЕТУ ЭЛЕКТРОННОЙ СТРУКТУРЫ МНОГОЗОННЫХ СИСТЕМ С СИЛЬНЫМИ КУЛОНОВСКИМИ КОРРЕЛЯЦИЯМИ

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Расчеты электронной структуры сильнокоррелированных систем, выполненные "slave-boson"-методом, сопоставлены с точными численными решениями для конечных кластеров. Показано, что "slave-boson"-метод является очень хорошим приближением при исследовании не только однозонной, но и многозонных моделей Хаббарда, позволяя определить характеристики основного состояния с точностью $\sim 1\%$. Выявлены причины обнаруженного в [6] сильного расхождения "slave-boson"-метода с точным решением.

В последние годы значительное внимание уделяется исследованию сильнокоррелированных соединений, к которым относятся, в частности, высокотемпературные сверхпроводники (ВТСП), системы с тяжелыми фермионами, магнитные полупроводники и пр. Ввиду исключительной сложности теоретического описания электронной структуры таких соединений (даже на основе упрощенных модельных гамильтонианов) в литературе обычно используются различные приближенные подходы, поскольку точные решения удается получить только численно и лишь для небольших кластеров [1, 2].