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REACTIVE FORCE ON A MOVING VORTEX FROM THE ADIABATIC SPECTRAL FLOW

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We calculate the contribution to the parameter D' of the reactive force acting on a moving vortex, which arises from the spectral flow of localized gapless fermions. Our consideration is valid in the regime $\omega_0\tau \ll 1$ and also limited to the adiabatic case $v_L k_F \tau \ll 1$ since we use the stationary density matrix. We assume discrete equidistant fermion spectra and show that the answer is the same as that obtained by Volovik in the continuous limit, namely, $D'_{\text{spec.flow}} = mk_F^3/6\pi^2$ per each anomalous branch. The result applies for the temperatures $T \ll T_c$. A possible extension to the temperatures close to T_c is suggested.

Introduction. Low-lying fermion modes localized inside cores of quantized vortices essentially affect low temperature thermodynamic and kinetic properties of superconductors and Fermi-superfluids [1]. Of especial interest are anomalous fermion branches which intersect the zero energy level as functions of approximate continuous quantum numbers, and pass very close to zero as functions of actual discrete quantum numbers. The characteristic energy scale of the lowest levels and the interlevel spacings on anomalous branches $\omega_0 = \Delta^2/E_F$ [2] is much smaller than the characteristic energy $\Delta \approx T_c$ for bulk fermions, therefore, effects of localized anomalous excitations become crucial at temperatures $T \ll T_c$. The hydrodynamic regime is determined by the inequality $\omega_0\tau \ll 1$, where τ is the effective time of quasiparticle relaxation. This will be our first restriction.

An important set of kinetic coefficients is related to the vortex motion. Consider a straight vortex line directed along the z axis, which moves uniformly as a whole with the velocity $\mathbf{v}_L \perp \hat{z}$. Then there is no net force acting on the vortex:

$$\rho_s(\mathbf{v}_s - \mathbf{v}_L) \times \vec{\kappa} + D(\mathbf{v}_n - \mathbf{v}_L) - D'(\mathbf{v}_n - \mathbf{v}_L) \times \vec{\kappa} = 0 \tag{1}$$

(see, e.g., [3]). Here $\vec{\kappa} = \hat{z}N\pi\hbar/m_3$ is the vector of circulation; $\mathbf{v}_s$, $\mathbf{v}_n$ are the superfluid and normal velocities. Parameters D, D' were calculated for conventional superconductors [4]; the same approach was also applied for superfluid ^{3}He [5–7]. In [8] a new treatment of the reactive force coefficient D' was proposed, based on the notion of the spectral flow along anomalous branches of localized fermions. The reactive force is given by the rate of the momentum creation due to the spectral flow from vacuum. In [8] the actual discrete fermionic spectrum was replaced by a continuous one, which is justified if the level width exceeds the interlevel spacing. The following answer for the contribution to the coefficient D' from one anomalous branch was obtained:

$$D'_{\rm spec.flow} = m\frac{k_F^3}{6\pi^2}. \quad (2)$$

We will show that in a simple model, in which only one anomalous branch with linear spectrum is taken into account, the same result holds in the adiabatic approximation for all temperatures $T < T_c$. We will also briefly discuss a possible extension to the realistic nonlinear spectrum.

For the sake of simplicity we shall restrict ourselves to axisymmetric vortices. In this case the relevant quantum numbers of fermionic excitations are the projection of the momentum on the z-axis, k_z, and the angular momentum projection, Q. Quasiclassical approximation, valid for $Q \gg 1$, gives an equidistant spectrum, which is linear in Q for each value of k_z [2, 5]:

$$E(Q, k_z) = Q\omega(k_z). \quad (3)$$

The interlevel spacing $\omega(k_z) \sim \Delta^2/E_F$ exhibits slow dependence on k_z [2], which will be irrelevant for us. The angular quantum number $Q = n + \nu$, $n \in Z$ can assume either integer ($\nu = 0$) or half-integer ($\nu = 1/2$) values [9].

Simple model for the momentum flow. Fermionic excitations are eigenmodes of the Bogolubov Hamiltonian. In order to build the model we replace it by a much simpler Hamiltonian

$$\hat{\mathcal{H}}_0 = \hat{Q}\omega(k_z), \quad (4)$$

where $\hat{Q}$ is the angular momentum operator of a fermion. Since we are going to study processes of creation from vacuum, the choice of the true vacuum reference frame becomes important. We postulate that the true frame is given by the normal component, i.e. moves with the velocity $\mathbf{v}_n$. We also suppose that the fermions born from the vacuum immediately get thermalized, and, therefore, the whole fermion system can be ascribed some temperature $T = 1/\beta$. The latter assumption is always justified in the regime $\omega(k_z)\tau \ll 1$.

Without any loss of generality we can put $\mathbf{v}_n = 0$. According to the general rule, the angular momentum of a moving fermion acquires the term $\mathbf{r} \times \mathbf{P}$, where $\mathbf{P}$ is the fermion's momentum. Taking into account that $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}_L$ and omitting the additive constant we get the following Hamiltonian in the laboratory frame [10]:

$$\hat{\mathcal{H}} = \left(\hat{Q} + t(\ \mathbf{v}_L \times \hat{\mathbf{P}}) \cdot \hat{z}\right)\omega(k_z). \quad (5)$$

The operator $\hat{Q}$ is diagonal in the Q-representation, while nonzero matrix elements of the momentum $\hat{\mathbf{P}}$ correspond to transitions $Q \to Q \pm 1$:

$$\hat{\mathbf{P}}_{QQ'} = \frac{1}{2}q\ \left((\hat{y} + i\hat{x})\delta_{Q,Q'+1} + (\hat{y} - i\hat{x})\delta_{Q,Q'-1}\right), \quad (6)$$

where q is the perpendicular momentum, $q^2 = k_F^2 - k_z^2$. We choose the axis x along the vortex velocity $\mathbf{v}_L$ and rewrite the operators of momentum projections and the Hamiltonian (5) in the second quantized form through fermionic operators a_n^+, a_n $(n \in Z)$:

$$\hat{P}_x = \frac{i}{2} q \sum_n (a_{n+1}^+ a_n - a_n^+ a_{n+1}), \quad (7)$$

$$\hat{P}_y = \frac{1}{2} q \sum_n (a_{n+1}^+ a_n + a_n^+ a_{n+1}), \quad (8)$$

$$\hat{\mathcal{H}}(t) = \sum_n \varepsilon_n a_n^+ a_n + \lambda(t) \sum_n (a_{n+1}^+ a_n + a_n^+ a_{n+1}), \quad (9)$$

where $\varepsilon_n = (n + \nu)\omega(k_z)$, $\lambda(t) = (1/2)\omega(k_z) q v_L t$.

We will use the equilibrium density matrix

$$\rho(t) = \exp(-\beta \hat{\mathcal{H}}(t)), \quad (10)$$

containing time only as a parameter entering $\lambda(t)$ (adiabatic regime). This is justified if $d\lambda/dt \ll \omega(k_z)/\tau$, which gives us the second restriction $v_L k_F \tau \ll 1$.

The rate of momentum creation is obtained as follows:

$$\partial_t \mathbf{P} = \partial_t \left(\mathrm{Tr}[\rho(t)\hat{\mathbf{P}}] / \mathrm{Tr}\, \rho(t) \right). \quad (11)$$

The expected value of the momentum can be expressed in terms of Matsubara Green functions,

$$G_{nm}(\tau_1 - \tau_2) = -\langle T_\tau a_n^M(\tau_1) \bar{a}_m^M(\tau_2) \rangle, \quad (12)$$

in the following way:

$$P_x(t) = (iq/2\beta) \sum_{n,k} (G_{n,n+1}(\omega_k) - G_{n+1,n}(\omega_k)), \quad (13)$$

$$P_y(t) = (q/2\beta) \sum_{n,k} (G_{n,n+1}(\omega_k) + G_{n+1,n}(\omega_k)). \quad (14)$$

Here $\omega_k = (\pi/\beta)(2k+1)$ are fermionic Matsubara frequencies.

One can easily see that $G_{n,n+1}(\omega) = G_{n+1,n}(\omega)$, which entails $P_x(t) \equiv 0$ as it should be because the reactive force is applied in the perpendicular to $\mathbf{v}_L$ direction. It can be shown (see below) that only the first term of the perturbation series in λ gives a nonzero contribution to $P_y(t)$:

$$P_y(t) = -q \frac{\lambda}{\omega(k_z)} = -\frac{q^2 v_L}{2} t. \quad (15)$$

The total reactive force from one anomalous branch is given by

$$F_y = \frac{1}{2} \int_{-k_F}^{k_F} \partial_t P_y(t) \frac{dk_z}{2\pi} = -\frac{v_L k_F^3}{6\pi}, \quad (16)$$

leading to (2). (The factor 1/2 removes the double counting of particle and whole momenta.) Thus, the equation (2) appears to be temperature independent in our simple model with one linear chain of equidistant levels.

A few technical details. The unperturbed Green function in the frequency representation reads:

$$G^{(0)}_{nm}(\omega) = \delta_{nm}\frac{1}{i\omega - \varepsilon_n}. \tag{17}$$

We denote $\mathcal{G}_n = 1/(i\omega - \varepsilon_n)$. The diagrammatic technique for the Hamiltonian (9) is as follows. In order to get the correction of the order $\lambda^{|m-n|+2s}$ $(s \geq 0)$ to the Green function $G_{nm}(\omega)$ one has to sum over all $\binom{|m-n|+2s}{s}$ paths Π_s of the lengths $|m-n|+2s$ that join n and m. The contribution of each path is proportional to the product of $|m-n|+2s+1$ zero Green functions $\mathcal{G}_k$ over all sites k visited by the path, including end points, multiplicities $\Pi_s(k)$ being taken into account:

$$G^{(s)}_{nm}(\omega) = \lambda^{|m-n|+2s} \sum_{\substack{\Pi_s \\ \partial\Pi_s=\{n,m\}}} \prod_k \mathcal{G}_k^{\Pi_s(k)}. \tag{18}$$

It follows that $G_{nm}(\omega) = G_{mn}(\omega)$. All multiple poles in the right hand side of (18) cancel out, leading to the result:

$$G^{(s)}_{nm}(\omega) = \lambda^{m-n+2s}\binom{m-n+2s}{s}\mathcal{G}_{n-s}\mathcal{G}_{n-s+1}\cdots\mathcal{G}_{m+s}, \quad n \leq m. \tag{19}$$

The equation (19) can be checked by a direct substitution into the Dyson equation,

$$G_{nm} = \delta_{nm}\mathcal{G}_n + \lambda\mathcal{G}_n(G_{n-1,m} + G_{n+1,m}), \tag{20}$$

which is satisfied in all orders.

According to (14) the lowest order contribution to the momentum creation is proportional to

$$\sum_{n,k} G^{(1)}_{n,n+1}(\omega_k) = \lambda\sum_{n,k}\mathcal{G}_n(\omega_k)\mathcal{G}_{n+1}(\omega_k). \tag{21}$$

Successively performing summations over frequencies and over sites we get

$$\lambda\sum_{n,k}\frac{1}{(i\omega_k - \varepsilon_n)(i\omega_k - \varepsilon_{n+1})} = -\lambda\beta/\omega(k_z), \tag{22}$$

which leads to (15). One can easily see that a more general relation is valid:

$$\lambda\sum_{n,k}\mathcal{G}_{n+r_1}(\omega_k)\mathcal{G}_{n+r_2}(\omega_k) = -\lambda\beta/\omega(k_z) \tag{23}$$

for all $r_1 \neq r_2$.

Higher corrections are expressed through sums of products of more than two Green functions. The relation

$$\mathcal{G}_p\mathcal{G}_q = \frac{\mathcal{G}_p - \mathcal{G}_q}{\omega(k_z)(p-q)}, \tag{24}$$

enables us to rewrite a product of N zero Green functions as a linear combination of products of $(N-1)$ Green functions. (The absence of multiple poles in (19) is essential here.) Applying the decomposition rule (24) several times we can express all higher order corrections linearly in products of triplets of zero Green functions:

$$\sum_{n,k}\mathcal{G}_{n+r_1}(\omega_k)\mathcal{G}_{n+r_2}(\omega_k)\mathcal{G}_{n+r_3}(\omega_k), \quad r_1 \neq r_2 \neq r_3. \tag{25}$$

We only have to demonstrate that (25) vanishes. In order to do this it is sufficient to decompose the product of any two of the three Green functions with the aid of (24) and then make use of (23).

We have shown that the adiabatic value of the spectral flow contribution to the reactive force coefficient D' is the same for a discrete equidistant spectrum and for a continuous spectrum in the regime $\omega_0\tau \ll 1$. For a singly quantized vortex possessing two anomalous branches this contribution equals $D'_{\text{spec.flow}} = mk_F^3/3\pi^2$. This leads to a reactive force which is linear in the velocity of the vortex. (The calculation of non-linear corrections requires the exact non-adiabatic treatment of the density matrix.) Moreover, we have found that in the simple model with the linear spectrum the spectral flow contribution does not depend upon T. Another contribution to the coefficient D' originates from the Iordanskii force [11, 10]: $D'_{\text{Iordanskii}} = -\rho_n$. The net value of the coefficient $D' = D'_{\text{spec.flow}} + D'_{\text{Iordanskii}} \approx \rho - \rho_n = \rho_s$ is close to that obtained in [3, 7].

Actually, our discussion was limited to the case of small temperatures $T \ll T_c$ because of two main reasons:

1) we used the linear form (3) of the anomalous branch spectrum, which is not applicable at temperatures close to T_c (in fact, the spectrum tends to $\pm\Delta$ for $Q \to \infty$);

2) we did not take into account the transitions that couple the anomalous branch with the other branches and extended states.

It can be shown that only the asymptotic values of the spectrum at $Q \to \infty$ enter the expression for the contribution of the anomalous branch to D':

$$D'_{\text{loc.spec.flow}}(T) = D'_{\text{spec.flow}} \cdot \tanh(\Delta/2T). \tag{26}$$

In order to get an agreement with [7], we should suppose that the temperature dependent factor in (26) is canceled by that arising from the transitions involving the other branches and extended states:

$$D'_{\text{ext.spec.flow}}(T) = D'_{\text{spec.flow}} \cdot (1 - \tanh(\Delta/2T)), \tag{27}$$

but we do not possess any direct demonstration of (27).

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