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QUANTUM DEFORMATIONS OF MULTI-INSTANTON SOLUTIONS IN THE TWISTOR SPACE

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We consider the quantum-group self-duality equation in the framework of the gauge theory on a deformed twistor space. Quantum deformations of the Atiyah–Drinfel'd–Hitchin–Manin and t'Hooft multi-instanton solutions are constructed.

The quantum-group gauge theory was considered in the framework of the algebra of local differential complexes [1 – 3] or as a noncommutative generalization of the fibre bundles over the classical or quantum basic spaces [4, 5].

We prefer to use local constructions of the noncommutative connection forms or gauge fields as a deformed analogue of the local gauge fields. In particular, the quantum-group self-duality equation (QGSDE) has been considered in the deformed 4-dimensional Euclidean space, and an explicit formula for the corresponding one-instanton solution has been constructed [3]. This solution can be treated as q-deformation of the BPST-instanton [6]. We shall discuss here quantum deformations of the general multi-instanton solutions [7].

The conformal covariant description of the classical (ADHM) solution was considered in Ref.[8]. We shall study the quantum deformation of this version of the twistor formalism. It is convenient to discuss firstly the deformations of the complex conformal group $GL(4,C)$, complex twistors and the complex linear gauge groups.

Let R^{ab}_{cd} $(a,b,c,d\ldots = 1\ldots 4)$ be the solution of the 4D Yang–Baxter equation satisfying also the Hecke relation

$$R\,R'\,R = R'\,R\,R', \tag{1}$$

$$R^2 = I + (q - q^{-1})R \tag{2}$$

where q is a complex parameter. Note that the standard notation for these R – matrices is $R = \hat{R}_{12},\ R' = \hat{R}_{23}$ [9].

Consider also the $SL_q(2,C)$ R-matrix

$$R^{\alpha\beta}_{\mu\nu} = q\delta^\alpha_\mu\delta^\beta_\nu + \varepsilon^{\alpha\beta}(q)\varepsilon_{\mu\nu}(q) \tag{3}$$

where $\varepsilon(q)$ is the deformed antisymmetric symbol.

Noncommutative twistors were considered in Ref.[10]. We shall use the R-matrix approach to define the differential calculus on the deformed twistor space.

Let z^α_a and dz^α_a be the components of the q-twistor and their differentials

$$R^{\alpha\beta}_{\mu\nu}\, z^\mu_a\, z^\nu_b = z^\alpha_c\, z^\beta_d\, R^{dc}_{ba}, \tag{4}$$

$$z^\alpha_a\, dz^\beta_b = R^{\alpha\beta}_{\mu\nu}\, dz^\mu_c\, z^\nu_d\, R^{dc}_{ba}, \tag{5}$$

$$dz^\alpha_a\, dz^\beta_b = -R^{\alpha\beta}_{\mu\nu}\, dz^\mu_c\, dz^\nu_d\, R^{dc}_{ba}. \tag{6}$$

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One can define also the algebra of partial derivatives ∂^a_α

$$R^{ab}_{cd}\,\partial^c_\alpha\,\partial^d_\beta = \partial^a_\mu\,\partial^b_\nu\,R^{\nu\mu}_{\beta\alpha}, \tag{7}$$

$$\partial^a_\alpha\,z^\beta_b = \delta^a_b\,\delta^\beta_\alpha + R^{\beta\mu}_{\alpha\nu}\,R^{da}_{cb}\,z^\nu_d\,\partial^c_\mu. \tag{8}$$

Consider the 4D deformed ε_q-symbol

$$R^{ba}_{fe}\,\varepsilon^{efcd}_q = -\frac{1}{q}\varepsilon^{abcd}_q. \tag{9}$$

The q-twistors satisfy the following identity:

$$\varepsilon^{abcd}_q\,z^\beta_b\,z^\mu_c\,z^\nu_d = 0. \tag{10}$$

The $SL_q(2)$-invariant bilinear function of twistors has the zero length in the projective 6D vector space

$$y_{ab} = \varepsilon_{\alpha\beta}(q)\,z^\alpha_a\,z^\beta_b = [P^{(-)}]^{dc}_{ba}\,y_{cd}, \tag{11}$$

$$(y,y) = \varepsilon^{abcd}_q\,y_{ab}\,y_{cd} = 0. \tag{12}$$

Consider a duality transformation $*$ of the basic differential 2-forms [3]

$$*\,dz\,dz' = dz\,dz'\,P^{(+)} - dz\,dz'\,P^{(-)}, \tag{13}$$

where $P^{(\pm)}$ are the projection operators of $GL_q(4)$ [9]. Note that the self-dual part $dz\,dz'\,P^{(+)}$ is proportional to

$$\varepsilon_{\alpha\beta}(q)\,dz^\alpha_a\,dz^\beta_b. \tag{14}$$

Let T^i_k be matrix elements of the $GL_q(N)$ quantum group

$$R_G T T' = T T' R_G, \tag{15}$$

where R_G is the R-matrix of $GL_q(N)$.

Quantum deformation of the $GL_q(N)$ gauge connection can be treated in terms of the noncommutative algebra for the components A^i_k of the connection 1-form [1, 2]

$$(A\,R_G\,A + R_G\,A\,R_G\,A\,R_G)^{ikl}_{mnp} = 0, \tag{16}$$

where $i,k,l,m,n,p = 1\ldots N$. These relations generalize the anticommutativity conditions for components of the classical connection form.

The restriction on the quantum trace of the connection $\alpha = \mathrm{Tr}_q A = 0$ is inconsistent with (16), but we can use the gauge-covariant relations $\alpha^2 = 0$, $\mathrm{Tr}_q A^2 = 0$ and $d\alpha = 0$ [3]. The curvature 2-form $F = dA - A^2$ is q-traceless for this model.

Consider the explicit realization of this gauge algebra in terms of z, dz and the set B of additional noncommutative parameters

$$A^i_k(z,dz,B) = dz^\alpha_a\,A^{ai}_{\alpha k}(z,B). \tag{17}$$

The analogous realizations were considered on the $GL_q(2)$ and $E_q(4)$ quantum spaces [1–3]. We shall treat the representation (17) as a local gauge field on the q-twistor space.

Let us consider the quantum deformation of the $GL(2)$ t'Hooft solution [8]:

$$A^{\alpha}_{\beta} = q^{-3} dz^{\alpha}_{a}\ (\partial^{a}_{\mu}\Phi)\Phi^{-1}\varepsilon^{\sigma\mu}(q)\varepsilon_{\sigma\beta}(q), \tag{18}$$

$$\Phi = \sum_{i}(X^{i})^{-1}, \qquad X^{i} = (y, b^{i}) = \varepsilon_{q}^{abcd} y_{ab}\ b^{i}_{cd}, \tag{19}$$

where b^{i}_{cd} are the noncommutative isotropic 6D vectors

$$db^{i}_{cd} = 0, \qquad (b^{i}, b^{i}) = 0, \tag{20}$$

$$[y_{ab}, X^{i}] = [b^{i}_{cd}, X^{i}] = 0. \tag{21}$$

The central elements X^{i} of the (B, z)-algebra do not commute with dz:

$$X^{i}\ dz^{\alpha}_{a} = q^{2} dz^{\alpha}_{a}\ X^{i}. \tag{22}$$

Stress that A^{α}_{β} satisfies Eq(16) and its quantum trace is a $U(1)$-gauge field with the zero field-strength

$$\mathrm{Tr}_{q} A = -q^{-3} d\Phi\Phi^{-1}, \qquad \mathrm{Tr}_{q} dA = 0. \tag{23}$$

The QGSDE for A^{α}_{β} is equivalent to the finite-difference Laplace equation for the function Φ on the q-twistor space

$$\Delta^{ba}\Phi(X^{i}) = \sum_{i}\Delta^{ba}\frac{1}{X^{i}} = 0, \tag{24}$$

$$\Delta^{ba}\Phi = \frac{q}{1+q^{2}}\varepsilon^{\alpha\beta}(q)\partial^{b}_{\beta}\partial^{a}_{\alpha}\Phi = (\partial^{ba} + \frac{1}{2}y_{cd}\partial^{dc}\partial^{ba})\Phi, \tag{25}$$

$$\partial^{ba} y_{cd} = [P^{(-)}]^{ab}_{dc}, \qquad \partial^{ba}(X^{i})^{-1} = -q^{-2}(X^{i})^{-2}(b^{i})^{ab}. \tag{26}$$

The ADHM-twistor functions of Ref.[7] can be connected with some $GL(N+2k)$ matrix function. Let us introduce the notation for indices of different types: $I, K, L, M = 1 \dots N+2k$ and $A, B = 1 \dots k$. The Ansatz for the general self-dual $GL_{q}(N, C)$ field contains the deformed twistors $u(z)$ and $\tilde{u}(z)$

$$A^{i}_{k} = du^{i}_{I}\tilde{u}^{I}_{k}, \qquad u^{i}_{I}\tilde{u}^{I}_{k} = \delta^{i}_{k}. \tag{27}$$

The commutation relations for the u and $\tilde{u}$ twistors are

$$(R_{G})^{ik}_{lm} u^{l}_{I} u^{m}_{K} = u^{i}_{L} u^{k}_{M} \mathrm{R}^{LM}_{IK}, \tag{28}$$

$$\mathrm{R}^{KI}_{ML}\tilde{u}^{L}_{i}\tilde{u}^{M}_{k} = \tilde{u}^{I}_{l}\tilde{u}^{K}_{m}(R_{G})^{ml}_{ki}, \tag{29}$$

$$\tilde{u}^{I}_{l}(R_{G})^{li}_{mk} u^{m}_{K} = u^{i}_{L}\mathrm{R}^{IL}_{KM}\tilde{u}^{M}_{k}, \tag{30}$$

where the R-matrices for $GL_{q}(N, C)$ and $GL_{q}(N+2k, C)$ are used.

Consider also the linear twistor functions v and $\tilde{v}$

$$v^{A\alpha}_{I} = z^{\alpha}_{a} b^{aA}_{I}, \tag{31}$$

$$\tilde{v}^{IA\alpha} = \tilde{b}^{IAa} z^{\alpha}_{a}. \tag{32}$$

Introduce the following condition for these functions:

$$v_I^{A\alpha}\tilde{v}^{IB\beta} = g^{AB}(z)\varepsilon^{\alpha\beta}(q), \tag{33}$$

where $g(z)$ is the nondegenerate $(k \times k)$ matrix with the central elements

$$g^{AB}(z) = \frac{q}{1+q^2} b_I^{aA}\, \tilde{b}^{IBb}\, y_{ab}. \tag{34}$$

The condition (33) is equivalent to the restriction on the elements of the B-algebra

$$[P^{(+)}]_{ab}^{cd}\, b_I^{aA}\, \tilde{b}^{IBb} = 0. \tag{35}$$

Write the basic commutation relations of the B-algebra

$$R_{cd}^{ab}\, b_I^{cA}\, b_K^{dB} = b_L^{aB}\, b_M^{bA}\, \mathrm{R}_{KI}^{ML}, \tag{36}$$

$$\mathrm{R}_{LM}^{IK}\, \tilde{b}^{LAa}\, \tilde{b}^{MBb} = R_{cd}^{ab}\, \tilde{b}^{IBc}\, \tilde{b}^{KAd}, \tag{37}$$

$$R_{cd}^{ab}\, b_I^{cA}\, \tilde{b}^{KBd} = \mathrm{R}_{IM}^{KL}\, \tilde{b}^{MBa}\, b_L^{aB}. \tag{38}$$

Remark that a formal permutation of the indices A and B is commutative. It is not difficult to define the relations between $b, \tilde{b}$ and z, dz.

Consider the new functions

$$\tilde{v}_{A\alpha}^I = g_{AB}(z)\varepsilon_{\alpha\beta}(q)\tilde{v}^{IB\beta} \tag{39}$$

where we use the inverse matrix with respect to the matrix (34).

Now one can construct the full quantum $GL_q(N+2k, C)$ matrices

$$\mathrm{U} = \begin{pmatrix} u_I^i \\ v_I^{A\alpha} \end{pmatrix}, \qquad \mathrm{U}^{-1} = \begin{pmatrix} \tilde{u}_i^I \\ \tilde{v}_{A\alpha}^I \end{pmatrix}. \tag{40}$$

The standard $GL_q(N+2k, C)$ commutation relations for these matrices contain Eqs.(28)–(30) and the relations for the v and $\tilde{v}$ functions.

Write explicitly the orthogonality and completeness conditions for the deformed ADHM-twistors

$$u_I^i\, \tilde{v}^{IA\alpha} = 0, \tag{41}$$

$$v_I^{A\alpha}\, \tilde{u}_i^I = 0, \tag{42}$$

$$\delta_K^I = \tilde{u}_i^I\, u_K^i\; + \; \tilde{v}^{IA\alpha}\, g_{AB}(z)\varepsilon_{\alpha\beta}(q)\, v_K^{B\beta}. \tag{43}$$

The gauge-field algebra (16) for the deformed ADHM-Ansatz (27) can be generated by the differential algebra on the $GL_q(N+2k, C)$ matrices U, U^{-1}, $d\mathrm{U}$ which contains the following relations:

$$\tilde{u}_i^I\, (R_G)_{lm}^{ik}\, du_K^l = du_L^k\, (\mathrm{R}^{-1})_{KM}^{IL}\tilde{u}_m^M, \tag{44}$$

$$du_L^i\, du_M^k\, (\mathrm{R}^{-1})_{IK}^{LM} = -(R_G^{-1})_{lm}^{ik}\, du_I^l\, du_K^m. \tag{45}$$

These relations are consistent with the commutation relation (28)–(30).

The self-duality of the connection (27) follows from Eqs.(31), (32), (41)–(43),

$$dA_k^i - A_l^i\, A_k^l = du_I^i\, (\tilde{u}_l^I\, u_M^l\; - \; \delta_M^I) d\tilde{u}_k^M = -u_I^i\, \tilde{b}^{IAa}\, g_{AB}(z)\varepsilon_{\alpha\beta}(q)\, dz_a^\alpha\, dz_b^\beta\, b_M^{Bb}\, \tilde{u}_k^M. \tag{46}$$

This curvature contains the self-dual 2-form (14) only.

It should be stressed that all R-matrices of our deformation scheme satisfy the Hecke relation with the common parameter q. The other possible parameters of different R-matrices are independent. The case $q = 1$ corresponds to the unitary deformations ($R^2 = I$) of the twistor space and the gauge groups. It is evident that the trivial deformation of the z-twistors is consistent with the nontrivial unitary deformation of the gauge sector and vice versa.

The Euclidean conformal q-twistors are a representation of the $U^*(4) \times SU_q(2)$ group. The antiinvolution for these twistors has the following form:

$$(z_a^\alpha)^* = \varepsilon_{\alpha\beta}(q)\, z_b^\beta\, C_a^b, \tag{47}$$

where C is the charge conjugation matrix for $U^*(4)$. We can use the gauge group $U_q(N)$ in the framework of our approach.

An analogous construction can be considered for the real twistors and the gauge group $GL_q(N, R)$.

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