Pis'ma v ZhETF, vol.62, iss.7, pp.583 - 588

LONG–TIME RELAXATION OF CURRENT IN A 2D WEAKLY DISORDERED CONDUCTOR

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Submitted 12 September 1995

The long-time relaxation of the average conductance in a 2D mesoscopic sample is studied within the method recently suggested by Muzykantskii and Khmelnitskii and based on a saddle-point approximation to the supermatrix σ–model. The obtained far asymptotics is in perfect agreement with the result of renormalization group treatment by Altshuler, Kravtsov and Lerner.

In the recent paper [1], Muzykantskii and Khmelnitskii (MK) considered the relaxation phenomena in disordered conductors in the framework of the supersymmetric σ–model approach. They suggested a nice idea that the long-time asymptotics of the conductance $G(t)$ is governed by a non-trivial saddle point of the σ–model. Their original goal was[*] to reproduce in a more direct way the result of Altshuler, Kravtsov and Lerner (AKL) [2], who found the logarithmically normal (LN) "tail" in the time dispersion in two and $(2+\epsilon)$ dimensions. However, MK found in 2D a different, power-law decay for moderately large times. They put forward a hypothesis that the LN asymptotics could hold for longer times. Here I will show that this is indeed the case, and that this result can be obtained via the method developed by MK.

Following MK, I consider a 2D disk-shaped sample of a radius R. I will consider the unitary symmetry (broken time reversal invariance) in course of the calculations. For systems of the orthogonal and symplectic symmetries, the treatment is completely analogous, and I simply present the corresponding results in the end of the paper. The problem can be described by the σ–model with the action [3]

$$S=-\frac{\pi\nu}{4}\int d^2r\,\mathrm{Str}[D(\nabla Q)^2+2i\omega\Lambda Q]\ . \tag{1}$$

Here $Q(r)$ is 4×4 supermatrix field, D is the diffusion constant, ν the density of states, ω the frequency, Str denotes the supertrace, and $\Lambda=\mathrm{diag}(1,1,-1,-1)$. The saddle point equation of MK reads:

$$\Delta_L\theta+\gamma^2\sinh\theta=0\ , \tag{2}$$

where $\theta(r)$ is the "non-compact angle" parametrizing the σ-model field $Q(r)$, Δ_L is the Laplace operator and $\gamma^2=i\omega/D$. It should be supplemented by the boundary conditions on the boundary with leads

$$\theta|_{\mathrm{leads}}=0 \tag{3}$$

and on insulating boundary

$$\nabla_n\theta|_{\mathrm{insulator}}=0\ , \tag{4}$$

where ∇_n denotes the normal derivative.

We can consider the two leads attached to the disk boundary to be of almost semicircular shape, with relatively narrow insulating intervals between them. Then we can approximate the boundary conditions by using eq.(3) for all the boundary, as it was done by MK. In fact, in view of the logarithmic dependence of the saddle point action on R (see below), the result should not depend to the leading aproximation on the specific shape of the sample and the leads attached. With the rotationally invariant form of the boundary condition, the minimal action corresponds to the function θ depending on the radius r only. We get therefore the radial equation

$$\theta'' + \theta'/r + \gamma^2 \sinh\theta = 0 \ ; \qquad 0 \le r \le R \tag{5}$$

(the prime denotes the derivative d/dr) with the boundary conditions:

$$\theta(R) = 0 \ , \tag{6}$$

$$\theta'(0) = 0 \ . \tag{7}$$

The condition (7) follows from the requirment of analyticity of the field in the disk center.

Assuming that characteristic values of θ satisfy the condition $\theta \gg 1$ (we will find below the corresponding restriction on the time t), one can replace $\sinh\theta$ by $e^\theta/2$. Eq.(5) can be then easily integrated, and its general solution reads:

$$e^{\theta(r)} = \frac{4C_1^2}{\gamma^2}\frac{C_2 r^{C_1-2}}{(C_2 r^{C_1}+1)^2} \ , \tag{8}$$

with two integration constants C_1 and C_2. To satisfy the boundary condition (7), we have to choose $C_1 = 2$. Furthermore, the above assumption $\theta(0) \gg 1$ implies that $4C_2/\gamma^2 \gg 1$. Therefore, the second boundary condition (6) is satisfied if $C_2 \simeq (4/\gamma R^2)^2$, and the solution can be written in the form

$$e^{\theta(r)} \simeq [(r/R)^2 + (\gamma R/4)^2]^{-2} \ . \tag{9}$$

Using now the self-consistency equation of MK,

$$2\pi \int_0^R dr\, r(\cosh\theta - 1) = t/\pi\nu \ , \tag{10}$$

one finds $\gamma^2 = 8\pi^2\nu/t$. Finally, the action

$$S \simeq \pi^2 \nu D \int dr\, r(\theta'^2 - \gamma^2 e^\theta) \tag{11}$$

is equal on the saddle point (9) to

$$S \simeq 8\pi^2 \nu D \ln(t\Delta) \ , \tag{12}$$

where $\Delta = 1/\nu\pi R^2$ is the mean level spacing. Eq.(11) coincides exactly with the result of MK. This consideration is valid provided $\theta'(r) < l^{-1}$ on the saddle point solution, which is the condition of the applicability of the diffusion approximation (here l is the mean free path). In combination with the assumption $\theta(0) \gg 1$ this means that $1 \ll t\Delta \ll (R/l)^2$.

Now I consider the ultra-long-time region, $t \gg \Delta^{-1}(R/l)^2$. In order to support the applicability of the diffusion approximation, we should search for a function $\theta(r)$ minimizing the action with an additional restriction $\theta' \leq Al^{-1}$. Here A is a parameter of the order of unity, which can not be fixed within the diffusion approximation. We will see however that the saddle–point action depends on l through $\ln(R/l)$ only, and thus does not depend on A in the leading order, so that we can set $A = 1$. Since the derivative has a tendency to increase in the vicinity of $r = 0$, the restriction can be implemented via replacing the boundary conditions (7) by $\theta'(r_*) = 0$, where the parameter r_* will be specified below. The solution reads now:

$$e^{\theta(r)} = \frac{(r/R)^{C-2}}{[(r/R)^C + \frac{C+2}{C-2}(r_*/R)^C]^2} \; ; \quad r_* \leq r \leq R. \tag{13}$$

The function $\theta(r)$ is ment as being constant within the vicinity $|r| \leq r_*$ of the disk center. The condition $\theta' \leq l^{-1}$ yields $r_* \sim lC$. It is important to note that the result does not depend on details of the cut-off procedure. For example one gets the same results if one chooses the boundary condition in the form $\theta'(r_*) = 1/l$. The crucial point is that the maximum derivative θ' should not exceed $1/l$. The constant C is to be found from the self-consistency equation (10) which can be reduced to the following form:

$$\left(\frac{R}{r_*}\right)^C = \frac{2t}{\pi^2 \nu R^2} \frac{C^2}{C-2}. \tag{14}$$

Neglecting corrections of the $\ln(\ln \cdot)$ form, we find

$$C \simeq \frac{\ln(t\Delta)}{\ln(R/r_*)} \simeq \frac{\ln(t\Delta)}{\ln(R/l)}. \tag{15}$$

The action (11) is then equal to

$$S \simeq \pi^2 \nu D (C+2)^2 \ln(R/r_*) \simeq \pi^2 \nu D \frac{\ln^2[t\Delta(R/l)^2]}{\ln(R/l)}. \tag{16}$$

For the orthogonal and symplectic ensembles, the saddle-point equation (5) has the same form, with the only difference that the action (11) is multiplied by the factor $\beta/2$, where $\beta = 1, 2, 4$ for the orthogonal, unitary and symplectic symmetries respectively. Combining eqs.(11) and (16), we get thus for the long-time asymptotics of the average conductance $G(t) \sim e^{-S}$ in all three symmetry cases:

$$G(t) \sim (t\Delta)^{-2\pi\beta g} \; , \qquad 1 \ll t\Delta \ll (R/l)^2; \tag{17}$$

$$G(t) \sim \exp\left\{-\frac{\pi\beta g}{4} \frac{\ln^2(t/g\tau)}{\ln(R/l)}\right\} \; , \qquad t\Delta \gg (R/l)^2, \tag{18}$$

where $g = 2\pi\nu D$ is the dimensionless conductance per square in 2D and τ is the mean free time.

The far asymptotical behavior (eq.(18)) is of the LN form and very similar to that found by AKL (see eq.(7.8) in Ref.[2]). It differs only by the factor $1/g$ in the argument of $\ln^2$. It is easy to see however that this difference disappears if one does the last step of the AKL calculation with a better accuracy. Let us consider for this purpose the intermediate expression of AKL (Ref.[2], eq.(7.11)):

$$G(t) \propto -\frac{\sigma}{\tau}\int_0^\infty e^{-t/t_\phi}\exp\left[-\frac{1}{4u}\ln^2\frac{t_\phi}{\tau}\right]\frac{dt_\phi}{t_\phi} \tag{19}$$

where $u \simeq \frac{1}{2\pi^2\nu D}\ln\frac{R}{l}$ in the weak localization region in 2D, which we are considering. Evaluating the integral (19) by the saddle point method, we find

$$\begin{aligned} G(t) &\sim \exp\left\{-\frac{1}{4u}\ln^2\frac{2ut}{\tau}\right\} , \\ G(t) &\sim \exp\left\{-\frac{\pi g}{4}\frac{\ln^2(t/g\tau)}{\ln(R/l)}\right\} , \end{aligned} \tag{20}$$

where we have kept only the leading term in the exponent. Eq.(20) is in *exact* agreement with eq.(18) for $\beta = 1$ (AKL assumed the orthogonal symmetry of the ensemble). Therefore, the supersymmetric treatment confirms the AKL result and also establishes the region of its validity. It is instructive to represent the obtained results in terms of the superposition of simple relaxation processes with mesoscopically distributed relaxation times t_ϕ:

$$G(t) \sim \int \frac{dt_\phi}{t_\phi} e^{-t/t_\phi} P(t_\phi) \ . \tag{21}$$

Then we have from eqs.(17), (18) for the distribution function $P(t_\phi)$:

$$P(t_\phi) \sim \begin{cases} (t_\phi/t_D)^{-2\pi\beta g} , & t_D \ll t_\phi \ll t_D\left(\frac{R}{l}\right)^2 \\ \exp\left\{-\frac{\pi\beta g}{4}\frac{\ln^2(t_\phi/\tau)}{\ln(R/l)}\right\} , & t_\phi \gg t_D\left(\frac{R}{l}\right)^2 , \end{cases} \tag{22}$$

where $t_D \simeq R^2/D$ is the time of diffusion through the sample.

For completeness, we list also the results for quasi-1D and 3D systems. For a quasi-1D sample (wire) of the length L (which is assumed to be much shorter than the localization length $\xi = 2\beta\pi\nu D$) the asymptotics read

$$G(t) \sim \exp\left\{-\frac{\beta\pi\nu D}{L}\ln^2(t\Delta)\right\} , \quad t\Delta \gg 1 \tag{23}$$

(for $\beta = 2$ this is just eq.(16) of MK). Eq.(23) is valid up to the exponentially large time $t \sim \exp(L/l)$ [1]. It is interesting to note that eq.(23) has essentially the same form as the asymptotical formula for $G(t)$ found by Altshuler and Prigodin [4] for the *strictly* 1D sample with a length much *exceeding* the localization length:

$$G(t) \sim \exp\left\{-\frac{l}{L}\ln^2(t/\tau)\right\} . \tag{24}$$

If we replace in eq.(24) the 1D localization length $\xi = 2l$ by the quasi-1D localization length $\xi = 2\beta\pi\nu D$, we reproduce the asymptotics (27) (up to a normalization of t in the argument of $\ln^2$, which does not affect the leading term in the exponent for $t \to \infty$). This leads us to make the following two conclusions. Firstly, this confirms once more the general conjecture [5] that the statistical properties of smooth envelopes of the wave functions in 1D and quasi-1D samples are identical. Secondly, this shows that the asymptotical "tail" (23) in the metallic sample is indeed due to "quasi-localized" eigenstates, as has been conjectured [1, 4, 6, 7].

In 3D, the analysis proceeds along the same line as for the ultra-long-time region in 2D. This is very similar to what has been done by MK in their consideration of the 3D case, so that I am presenting a brief sketch of the derivation only. I consider the saddle–point equation in the spherically symmetric form

$$\theta'' + 2\theta'/r + \gamma^2 \sinh\theta = 0 \ , \quad \theta(R) = 0 \ , \ \theta'(r_*) = 0. \tag{25}$$

At $r \gg r_*$ the last term in the l.h.s. of eq.(25) can be neglected [1] and $\theta(r) \simeq C(R/r - 1)$. The maximum derivative θ' is reached at $r \sim r_*$, so that the condition $\theta' l \leq A$ with $A \sim 1$ implies $r_* \sim (CRl)^{1/2}$. The self-consistency equation

$$4\pi \int r^2 dr(\cosh\theta - 1) = t/\pi\nu$$

then yields $r_*^3 e^{CR/r_*} \sim t/\nu$, and consequently,

$$C \sim \frac{l}{R} \ln^2 \left[\frac{t}{\tau(k_F l)^2}\right] \ ; \qquad r_* \sim l \ln \left[\frac{t}{\tau(k_F l)^2}\right]$$

Multiplying eq.(25) by θ' and integrating it from r_* to R, we get $\gamma^2 \exp\theta(r_*) \sim 1/l^2$. Finally, the saddle–point action is estimated as

$$\begin{aligned} S(t) \ &\simeq \ 2\pi^2 \nu D\beta \int r^2 dr(\theta'^2 - \gamma^2 e^\theta) \sim \\ &\sim \ \nu D \frac{r_*^3}{l^2} \sim \nu D l \ln^3 \left[\frac{t}{\tau(k_F l)^2}\right]. \end{aligned} \tag{26}$$

Since the action (26) is proportional to l, the numerical prefactor in (26) can not be found within the diffusion approximation. We get therefore the following contribution to the average conductance $G(t)$,

$$G(t) \sim \exp\{-S(t)\} \ , \qquad S(t) \sim (k_f l)^2 \ln^3 \left[\frac{t}{\tau(k_f l)^2}\right] \ , \tag{27}$$

which dominates over the usual Cooperon–induced term [1, 2], $G(t) \sim \exp(-t/t_D)$, at $t \gg (k_f l)^2 t_D \ln^3(t_D/\tau)$.

Let us discuss now the approximations involved, and the limitations of the method. We have employed the saddle–point method for the σ–model and found that for long enough time in 2D (and also in 3D) the solution violates in the vicinity of its "center" $r = 0$ the condition of applicability of the diffusion approximation, $\theta' l \ll 1$. We assumed then that the correct result can be found by allowing only those configurations of the Q–field, which satisfy the restriction $\theta' l \leq A$ with $A \sim 1$. This is a natural extension of cutting the diffuson and Cooperon momenta at $q \sim l^{-1}$, the procedure commonly used in perturbation theory. To implement the above restriction, we shifted the boundary condition from the singular point $r = 0$ to $r = r_*$. Though the numerical constant $A \sim 1$ can not be specified in this way, it does not affect the long-time behavior of $G(t)$ in 2D case. At the same time, it enters the prefactor (in front of the logarithm cube) in the exponent in 3D case. It would be desirable, of course, to justify rigorously the above cut-off procedure, and to fix the numerical coefficient in the action in 3D. For this purpose, one should go beyond the long-wave-length σ–model approximation and develop a generalization of the method valid also on a scale $r \lesssim l$. A step forward in this direction was done in a very recent paper [8].

We note in conclusion, that the obtained long-time asymptotics of the average conductance $G(t)$ have a form very similar to the asymptotical behavior of the distribution function $P(\rho)$ of local density of states (LDOS) [9]. In both cases, the result is of the LN form in quasi-1D and 2D, and of a somewhat different (though very similar) "log-cube-exponential" form in 3D. As in the case of LDOS distribution [9], we have found a perfect agreement with the result of renormalization group (RG) treatment [2] in 2D. I believe this agreement between the RG and supersymmetric treatments of $G(t)$ and $P(\rho)$ to be of considerable conceptual importance. It is of a non-trivial nature, since the supersymmetric solution heavily relies on the non-compact structure of the supersymmetric σ-model manifold and is dominated by the large values $\theta \gg 1$ of the "non-compact angle" θ, whereas the RG consideration is just a resummation of the perturbative expansion and does not distinguish between the compact and non-compact versions of the σ-model. This agreement provides strong support to other results obtained within the RG approach in the weak localization region and in the vicinity of the Anderson transition [2]. On the other hand, we see that the supersymmetry method is in many cases able to reproduce results of RG treatment in a more elegant way. Furthermore, it is not restricted like RG to the spatial dimension $d=2$ and can be successfully applied to quasi-1D and 3D systems as well. Besides the study of conductivity relaxation $G(t)$ and LDOS distribution $P(\rho)$ discussed above, I would like to mention in this context the recent progress in understanding of the statistical properties of eigenfunctions [5, 10, 11]. Seeing that the two approaches are in amazingly good agreement, we can (depending on the problem considered) use any of them or even combine them to complete our understanding of the properties of mesoscopic disordered systems.

I am grateful to V.E.Kravtsov and D.E.Khmelnitskii for valuable discussions and comments. This work was supported by SFB 195 der Deutschen Forschungsgemeinschaft.

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1. B.A.Muzykantskii and D.E.Khmelnitskii, Phys.Rev. **B 51**, 5480 (1995).
2. B.L.Altshuler, V.E.Kravtsov, and I.V.Lerner, In: *Mesoscopic Phenomena in Solids*, Ed. by B.L.Altshuler, P.A.Lee and R.A.Webb (North Holland, Amsterdam , 1991).
3. K.B.Efetov, Adv.Phys. **32**, 53 (1983).
4. B.L.Altshuler and V.N.Prigodin, Pis'ma Zh. Eksp. Teor. Fiz. **47**, 36 (1988) [JETP Lett. **47**, 43 (1988)].
5. Y.V.Fyodorov and A.D.Mirlin, Int. Journ. Mod. Phys. B **8**, 3795 (1994).
6. V.E.Kravtsov, *Habilitationsschrift*, Heidelberg, 1992.
7. B.L.Altshuler and V.N.Prigodin, Zh. Eksp. Teor. Fiz. **95**, 348 (1989) [Sov. Phys. JETP **68**, 198 (1989)].
8. B.A.Muzykantskii and D.E.Khmelnitskii, Pis'ma Zh. Eksp. Teor. Fiz. [JETP Lett.] (25 June 1995).
9. A.D.Mirlin, Preprint cond-mat/9507082, submitted to Phys. Rev. B.
10. Y.V.Fyodorov and A.D.Mirlin, Phys.Rev. **B 51**, 13403 (1995).
11. V.I.Fal'ko and K.B.Efetov, Preprints cond-mat/9503096, cond-mat/9507091.