Pis'ma v ZhETF, vol.62, iss.11, pp.839 - 843 © 1995 December 10

RADIATIVE MECHANISM OF LEPTON BOUND STATES PRODUCTION

A.I.Akhiezer, N.P.Merenkov[1)]

National Science Centre "Kharkov Institute of Physics and Technology"
310108 Kharkov, Ukraine

Submitted 2 November 1995

The radiative mechanism of lepton bound states formation is investigated.It concludes in a real photon emission and bound state production at free lepton interaction. The cross-sections for corresponding processes are derived, and some figures given for the case of positronium formation in free electron-positron collision.

The different bound states of opposite charged leptons may be produced due the electromagnetic interaction. The same interaction leads to the formation of the bound states of leptons with proton and antiproton. The theory of all these states bases on the Coulomb interaction and consequently it may be constructed by the same method.

We shall show the theory of such states formation first of all on the example of the positronium. Note that the quantum states and decay mechanisms of a positronium are well known in theory [1,2]. Moreover, the mechanism of the radiationless capture of the positron with an atomic electron is well investigated (see [3-6] and references therein). Meanwhile there is another mechanism of positronium formation, namely the mechanism of radiative capture. It describes the formation of the positronium in electron-positron collision with the resulting emission a real photon. In this case both the initial particles (electron and positron) are free. This mechanism is in the essence analogous to the radiative capture of a proton by a neutron (first investigated by Fermi [7]), which result in real photon emission and deuteron production. The study of this mechanism is of great interest, especially in connection with electron-positron plasma properties investigation [8].

The theoretical description of the radiative capture is straightforward as it is based on the general methods of quantum electrodynamics perturbation theory. We deal here with the radiative transition of particles from the free state to the bound one. Therefore this mechanism may be considered as a peculiar inverse photoeffect. It may be used for investigating the formation of the different bound states. In this paper we consider the mechanism of the radiative capture of leptons with arbitrary masses which results in a real photon radiation and bound state formation.

In Section 1 we give the formulae for corresponding matrix elements and cross-sections, and in Section 2 the case of positronium formation is investigated.

1. In this Section we consider the transition of the system containing two opposite charged free particles into the bound state. This transition must be accompanied with a real photon radiation. For a definition we shall speak about the transition electron and muon to muonium (but the all derived formulae can be applied to any radiative transition process of two opposite charged particles interacting by Coulomb law):

$$\mu^+(p_+) + e^-(p_-) \to \gamma(k) + (\mu^+e^-)(p), \tag{1}$$

[1)]e-mail: kfti@rocket.kharkov.ua

where corresponding 4-momenta are indicated in the brackets.

The differential probability of the radiative transition (1) related to the unit volume and the unit time may be written as follows

$$dw = \frac{dW}{TV} = \frac{e^2\omega d\Omega |\overline{M}|^2}{2(2\pi)^2}, \tag{2}$$

where ω is the energy and $d\Omega$ is the solid angle element of photon. The reduced matrix element which enters into Eq.(2) will be defined as

$$\overline{M} = \int d^3r \Psi_f^*(\mathbf{r}) \left(\frac{\mathbf{a}\nabla}{\mathrm{m}} e^{i\mathbf{k}_1\mathbf{r}} + \frac{\mathbf{a}\nabla}{\mu} e^{-i\mathbf{k}_2\mathbf{r}} \right) \Psi_i(\mathbf{r}), \tag{3}$$

where $\mathbf{r}$ is the radius-vector of the related motion, $\mu(m)$ is muon (electron) mass, $\mathbf{k}_1 = \mu\mathbf{k}/(\mu + \mathrm{m})$, $\mathbf{k}_2 = m\mathbf{k}/(\mu + m)$, $\nabla = \partial/\partial\mathbf{r}$. Wave functions $\Psi_i(\mathbf{r})$ and $\Psi_f(\mathbf{r})$ describe the relative motion of the electron and muon in the initial and final states, respectively. The photon energy ω is related to the initial electron momentum $\mathbf{p}_- \equiv \mathbf{p}$ and the binding energy of the muonium by

$$\omega + \frac{\omega^2}{2(m + \mu - \Delta)} = \Delta + \frac{p^2}{2\mu m}(m + \mu). \tag{4}$$

In the problem under consideration the wave functions Ψ_i and Ψ_f must be the eigenfunctions of the same hamiltonian. Since Ψ_f is the bound-state wave function, it is obvious the that the influence of the interaction on the wave function must be taken into account in the initial state too. This means that $\Psi_i(\mathbf{r})$ has the structure of the wave function of the continuous spectrum of the charged particle in the Coulomb field, and this ensures the orthogonality of $\Psi_i(\mathbf{r})$ and $\Psi_f(\mathbf{r})$. In the process (1) the free electron and muon exist only in the initial state, therefore the function $\Psi_i(\mathbf{r})$ at large r's must have the form a superposition of the plane wave and the outgoing spherical wave. The function with this asymptotic is [9]

$$\Psi_i(\mathbf{r}) = \exp(\pi\zeta/2)\Gamma(1 - i\zeta)F(i\zeta, 1, i(pr - \mathbf{pr}))e^{i\mathbf{pr}} \equiv \Psi(\mathbf{p}, \mathbf{r}), \tag{5}$$

where $\zeta = \alpha m\mu/p(m + \mu)$, $\alpha = 1/137$ is the fine structure constant, F is the hypergeometrical function. Note that the function $\Psi(\mathbf{p}, \mathbf{r})$. is normalized in the same way as the plane wave

$$\int \Psi^*(\mathbf{p}, \mathbf{r})\Psi(\mathbf{p}', \mathbf{r})d^3r = (2\pi)^3\delta(\mathbf{p} - \mathbf{p}')$$

We take the ground-state wave function of the muonium as

$$\Psi_f = \sqrt{\frac{\eta^3}{\pi}} e^{-r\eta}, \quad \eta = \frac{m\mu\alpha}{m + \mu} \equiv \zeta p, \tag{6}$$

where $1/\eta$ is the muonium radius.

Using wave function (5) and (6) we derive with the help of the Gauss theorem

$$\overline{M} = N \int d^3r e^{-r\eta} \frac{\mathbf{ar}}{r} \left(\frac{e^{-i\mathbf{k}_1\mathbf{r}}}{m} + \frac{e^{i\mathbf{k}_2\mathbf{r}}}{\mu} \right) e^{i\mathbf{pr}} F(i\zeta, 1, i(pr - \mathbf{pr})), \tag{7}$$

where

$$N = \sqrt{\eta^5/\pi}\exp(\pi\zeta/2)\Gamma(1 - i\zeta).$$

The integral which enters in (7) can be counted with the help of the following formula [10]

$$\int e^{i(\mathbf{p}-\mathbf{k})\mathbf{r}-r\eta}F(i\zeta, 1, i(pr - \mathbf{pr}))d^3r/r = 4\pi\frac{[k^2 + (\eta - ip)^2]^{-i\zeta}}{[(\mathbf{p}-\mathbf{k})^2 + \eta^2]^{1-i\zeta}}.$$

It is easy to show that

$$\overline{M} = i(1 - i\zeta)N8\pi(\mathbf{ap})[\frac{[\mathbf{k}_1^2 + (\eta - ip)^2]^{-i\zeta}}{m[(\mathbf{p}-\mathbf{k}_1)^2 + \eta^2]^{2-i\zeta}} + \frac{[\mathbf{k}_2^2 + (\eta - ip)^2]^{-i\zeta}}{\mu[(\mathbf{p}+\mathbf{k}_2)^2 + \eta^2]^{2-i\zeta}}]. \qquad (8)$$

When writing the last formula we took into account that $\mathbf{ak}_1 = \mathbf{ak}_2 = 0$.

The matrix element squared will be defined as follows

$$|\overline{M}|^2 = \frac{2^8\pi^2\zeta(1+\zeta^2)}{1 - e^{-2\pi\zeta}}(\mathbf{ap})^2\eta^5 A \quad , \qquad (9)$$

where

$$A = \frac{e^{-2\zeta\phi_1}}{m^2a_1^4} + \frac{e^{-2\zeta\phi_2}}{\mu^2a_2^4} + \frac{2\cos(\zeta(\ln(\frac{r_2}{r_1}r_{12}))}{m\mu a_1^2a_2^2}e^{-(\phi_1+\phi_2)},$$

$$a_1 = (\mathbf{p}-\mathbf{k}_1)^2 + \eta^2, \quad a_2 = (\mathbf{p}+\mathbf{k}_2)^2 + \eta^2, \quad \phi_{1,2} = \mathrm{arctg}\frac{2p\eta}{k_{1,2}^2 + \eta^2 - p^2},$$

$$r_{1,2} = |k_{1,2}^2 + (\eta + ip)^2|, \quad r_{12} = \frac{a_1}{a_2}.$$

It follows from (4) that the photon energy $\omega << p$ if the kinetic energy of the relative motion of the initial particles more or the same order as the muonium binding energy. Therefore the expression for A can be expanded in this limited case as

$$A = \frac{1}{(p^2+\eta^2)^4}\left\{\frac{(m+\mu)^2}{m^2\mu^2} + \frac{8p\cos\Theta}{p^2+\eta^2}\left(\frac{k_1}{m^2} - \frac{k_2}{\mu^2} + \frac{k_1-k_2}{m\mu}\right) + \right.$$

$$+\left[\frac{2}{p^2+\eta^2}\left(-2 + \frac{20p^2\cos^2\Theta}{p^2+\eta^2}\right) + \frac{4\eta^2}{(p^2+\eta^2)^2}\right]\left(\frac{k_1^2}{m^2} + \frac{k_2^2}{\mu^2}\right) +$$

$$+\frac{2}{m\mu}\left[\frac{k_1^2+k_2^2}{p^2+\eta^2}\left(-2 + \frac{12p^2\cos^2\Theta}{p^2+\eta^2}\right) - \frac{16p^2\cos^2\Theta}{(p^2+\eta^2)^2}k_1k_2\right] +$$

$$\left. +\frac{2}{m\mu}\left[\frac{2\eta^2(k_1^2+k_2^2)}{(p^2+\eta^2)^2} - \frac{2\eta^2\cos^2\Theta}{p^2+\eta^2)^3}(k_1^2+k_2^2)^2\right]\right\}, \qquad (10)$$

where Θ is the angle between vectors $\mathbf{p}$ and $\mathbf{k}$.

If the momentum of the relative motion of the iitial particles $p << \eta$ the corresponding expression wil be defined by the formula

$$A = \frac{e^{-4}}{\eta^8}\left[\frac{(m+\mu)^2}{m^2\mu^2}(1-\frac{8p^2}{3\eta^2}) + \frac{8p\cos\Theta}{\eta^2(m+\mu)}(\frac{k_1}{m}-\frac{k_2}{\mu}) - \frac{4(k_1+k_2)^2\cos^2\Theta}{m\mu\eta^2}\right]. \quad (10a)$$

When writing (11) we took into account that in this limited case the photon energy is of the order of the muonium binding energy: $\omega \cong \Delta \approx \alpha\eta \ll \eta$.

The differential probability (2) defines the differential cross-section $d\sigma = dw/v_r$, where $v_r = p(m+\mu)/m\mu$ is the relative velocity of the electron and muon in the initial state. (We suppose the particle densities equal to unit).

Thus, summing over photon polarizations we can write the differential cross-section of the process (1) in the case $p \geq \eta$ as follows

$$d\sigma = 2^7\pi\frac{\omega}{p}\zeta\left(\frac{\zeta^2}{1+\zeta^2}\right)^3\frac{e^{-4\mathrm{arcctg}\zeta}}{1-e^{-2\pi}}d\Omega\sin^2\Theta\times$$

$$\times\left\{\frac{(m+\mu)^2}{m^2\mu^2} + \frac{8\omega(\mu^2-m^2)\cos\Theta}{p(1+\zeta^2)m^2\mu^2} + \frac{4\omega^2}{p^2(1+\zeta^2)^2m^2\mu^2}\times\right.$$

$$\left.\times[(m^2+\mu^2)(10\cos^2\Theta-1) + m\mu(1-\cos^2\Theta(14+\zeta^2))]\right\}. \quad (11)$$

If the initial particles have equal masses : $\mu = m$ (as in the case of radiative transition of an electron and a positron to positronium) the second term in the brackets is absent.

The angular integration of the r.h.s. (12) gives the following expression for the total cross-section of the radiative capture of the electron by the muon

$$\sigma = \frac{2^{10}\pi^2\omega}{3pm^2\mu^2}\left(\frac{\zeta^2}{1+\zeta^2}\right)^3\frac{e^{-4\mathrm{arctg}\zeta}}{(1-e^{-2\pi\zeta})}\left[(m+\mu)^2 + \frac{4\omega^2(5(\mu-m)^2+m\mu(1-\zeta^2))}{5p^2(1+\zeta^2)^2}\right]. \quad (12)$$

Formula (13) is valid for the cross-section of the radiative capture opposite charged particles with arbitrary masses which interact by Coulomb law. It applied to the process of positronium formation at free electron and positron collision as well as to the muonium formation and to the capture of electron, muon and τ-lepton by proton.

2. In the case of radiative transition of an electron and a positron to the positronium we have

$$\sigma = \frac{2^{12}\pi^2\omega}{3pm^2}\zeta\left(\frac{\zeta^2}{1+\zeta^2}\right)^3\frac{e^{-4\mathrm{arctg}\zeta}}{(1-e^{-2\pi\zeta})}\left[1+\frac{\omega^2(1-\zeta^2)}{5p^2}\right], \quad \zeta = \frac{\alpha m}{2p}, \quad (13)$$

and the photon energy is defined from the energy conservation law :

$$\omega + \omega^2/4m = p^2/m + \alpha^2 m/4.$$

This formula is valid if $\zeta \leq 1$.

If the relative momentum of the initial particles is very small, and $\zeta \gg 1$ then the differential cross-section for positronium formation has a following form

$$\sigma = \frac{2^7\pi^2\alpha^3e^{-4}}{p^2}\sin^2\Theta\left[1-\frac{2p^2+12\omega^2\cos^2\Theta}{3\eta^2}\right]d\cos\Theta. \quad (14)$$

The angular integration of (14) gives the total cross-section

$$\sigma = \frac{2^9\pi^2\alpha^3 e^{-4}}{3p^2}\left[1 - \frac{10p^2+12\omega^2}{15\eta^2}\right].$$

We see that in the case of small initial electron momenta the cross-section is proportional to p^{-2} in contrast to the cross-section of the slow neutron capture by proton, which behaves in this case as p^{-1}. This is the consequence of the long-range Coulomb interaction as compared with short-cut strong interaction.

Note that the total cross-section σ of the positronium formation is derived without taking into account the particle spins. Consequently, in order to obtain singlet positronium production we must multiply it on 1/4. In the case of triplet positronium we have to sum over spin states of the positronium, and this gives a trebled cross-section value

$$\sigma^{\uparrow\downarrow} = \frac{1}{4}\sigma, \quad \sigma^{\uparrow\uparrow} = \frac{3}{4}\sigma.$$

Table lists the total cross-section values calculated by formula (14) for the rsdiative transition of the electron and positron to positronium at p-values ranging between 1 Kev/c and 100 Kev/c.

The total cross-section of the positronium formation

p(keV/c)	σ(cm^2)	p(keV/c)	σ(cm^2)
1	$3.92\cdot10^{-21}$	21.5	$4.34\cdot10^{-26}$
2.15	$5.2\cdot10^{-22}$	46.4	$1.08\cdot10^{-27}$
4.46	$3.69\cdot10^{-23}$	100	$2.41\cdot10^{-29}$
10	$1.48\cdot10^{-24}$		

The cross-section σ increases and reaches very large values as the energy of theinitial particles decreases. In consequence, process considered should be take into account on investigating the electron-positron plasma properties, as well as the low energy annihilation of electron and positron into photons.

The authors thank S.V.Peletminsky for useful discussions and critical remarks.

This work was supported, in part, by the International Soros Science Education Programm (ISSEP).

1. J.Pirenne, Arch. Sc. Phys. et Nat. **29**, 207 (1947); V.B.Berestetsky and L.D.Landau, Sov. JETP **19**, 673 (1949); V.B.Berestetsky, Sov. JETP **19**, 1130 (1949).
2. I.Pomeranchuk, DAN SSSR **60**, 218 (1947); A.Ore and J.L.Powell, Phys. Rev. **75**, 1696 (1949).
3. J.C.Cheng and P.J.Kramer, Phys. Rev. **A5**, 1207 (1972).
4. N.K.Sarkar, M.Basu, and A.S.Ghosh, Phys. Rev. **A45**, 6887 (1992); M.Basu and A.S.Ghosh, J. Phys. **B21**, 3439 (1988).
5. A.Jagarashi and N.Toshima, Phys. Rev. **A46**, R1159 (1992); N.C.Deb, J.H.Mc Guire, and N.C.Sil, Phys.Rev. **A36**, 1082 3707 (1987).
6. T.D.Steiger and R.S.Conti, Phys. Rev. **A45**, 2744 (1992); S.Tripathi and C.Sinha, Phys. Rev. **A42**, 5743 (1990).
7. E.Fermi, Phys. Rev. **48**, 570 (1935).
8. A.I.Akhiezer, S.V.Peletminsky, and A.A.Yatsenko, LTP **20**, 509 (1994).
9. A.Sommerfeld, *Atombau und Spektrallinien*. II Band. Frieder. Vieweg & Sonh. Braunschweig 1951.
10. A.I.Akhiezer and V.B.Berestetsky. *Quantum Electrodynamics*, Moscow Nauka, 1969.