

A NEW METHOD OF QUANTIZATION OF CLASSICAL SOLUTIONS

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Using stochastic quantization method [1], we derive equations for correlators of quantum fluctuations around the classical solution in the massless ϕ^4 theory. The obtained equations are then solved in the lowest orders of perturbation theory, and the first correction to the free propagator of a quantum fluctuation is calculated.

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1. *Introduction.* Recently new equations for vacuum correlators in the ϕ^3 theory and in gluodynamics were derived via stochastic quantization method [2]. These equations, alternative to the Schwinger-Dyson and Makeenko-Migdal equations, allow one to connect correlators with different number of fields, but, in contrast to the Schwinger-Dyson equations, they are gauge-invariant in the case of gauge theories, and their mathematical structure is simpler than the structure of Makeenko-Migdal equations.

In this letter we shall apply this approach to derivation of a set of equations for correlators of quantum fluctuations around the classical solution (lipaton) [3] in the massless $\lambda\phi^4$ theory. To this end we solve the Langevin equation for quantum fluctuation through Feynman-Schwinger path integral representation, and, introducing the generating functional $Z[J]$, defined by the formula (4), apply to it cumulant expansion [4,5]. The latter in this case has pure perturbative meaning and corresponds to the usual semiclassical method of quantization of classical solutions, suggested in [6] and developed in [7], providing expansion in the powers of coupling constant (or in the powers of Planck constant). The generating functional $Z[0]$, where one neglects for simplicity the second term in the exponent in (3), has a meaning of a one-loop expression for the effective potential, generated by quantum fluctuations. This analogy becomes clear after applying to $Z[0]$ cumulant expansion: an n -th term of cumulant expansion is an n -point Green function with n lipatonic insertions.

After that we expand the obtained system of bilocal approximation in the two lowest orders of perturbation theory and solve the obtained equations in the case, when the lipaton size is large enough (the particular meanings of this approximation for each of the equations of bilocal approximation are clarified in section 3). As a result we get the first constant correction to the free propagator of a quantum fluctuation, which occurs to be of the order of $1/\rho^2$, where ρ is a lipaton size.

Note, that stochastic quantization method was never applied before to quantization of classical solutions. We hope, that the suggested method will be especially useful in gauge theories, because, as was shown in [2], in this case it preserves gauge invariance. The corresponding equations for quantum corrections to an instanton will be treated elsewhere.

The plan of the letter is the following. In section 2 we derive a system of equations for correlators of quantum fluctuations in bilocal approximation. In section 3 we solve these equations up to the order of $\lambda^{1/2}$ under the assumption, that the lipaton size is large enough, and compute the first correction to the

free propagator of a quantum fluctuation. The main results of the letter are summarized in the Conclusion.

2. *Equations for quantum fluctuations in Gaussian approximation.* We shall start with the action of Euclidean massless $\lambda\phi^4$ theory, which has the form

$$S = \int dx \left[\frac{1}{2}(\partial_\mu\phi)^2 - \frac{\lambda}{4}\phi^4 \right], \quad \lambda > 0.$$

In this theory exists a classical solution (lipaton)

$$\phi_{cl}(x) = 2\sqrt{\frac{2}{\lambda}} \frac{\rho}{(x-x_0)^2 + \rho^2}$$

with a centre x_0 and of an arbitrary size ρ [3].

Let the lipaton with the centre $x_0 = 0$ be a classical part of the total field $\phi: \phi = \phi_{cl} + \varphi$, where φ is a quantum fluctuation in the lipatonic background. This splitting yields the following Langevin equation for φ :

$$\dot{\varphi} - (\partial^2 + 3\lambda\phi_{cl}^2 + 3\lambda\phi_{cl}\varphi + \lambda\varphi^2)\varphi = \eta,$$

where

$$\langle \eta(x, t)\eta(x', t') \rangle = 2\delta(x-x')\delta(t-t'), \quad (1)$$

$$\langle \dots \rangle \equiv \frac{\int D\eta(\dots) \exp\left[-\frac{1}{4} \int dx dt \eta^2(x, t)\right]}{\int D\eta \exp\left[-\frac{1}{4} \int dx dt \eta^2(x, t)\right]}.$$

Considering Langevin time as a proper time, one can solve this equation via Feynman-Schwinger path integral representation. The solution, corresponding to the retarded Green function, has the form:

$$\varphi(x, t) = \int_0^t dt' \int dy (Dz)_{xy} K_z(t, t') F_z(t, t') \eta(y, t'), \quad (2)$$

where

$$K_z(t, t') \equiv \theta(t-t') \exp \left\{ \int_{t'}^t d\xi \left[-\frac{\dot{z}^2(\xi)}{4} + 3\lambda\phi_{cl}^2(z(\xi)) \right] \right\}, \quad (Dz)_{xy} = \lim_{N \rightarrow \infty} \prod_{n=1}^N \frac{d^4 z(n)}{(4\pi\epsilon)^2},$$

$$N\epsilon = t-t', \quad z(\xi=t') = y, \quad z(\xi=t) = x$$

$$\text{and } F_z(t, t') \equiv \exp \left\{ \lambda \int_{t'}^t d\xi [3\phi_{cl}(z(\xi))\varphi(z(\xi), \xi) + \varphi^2(z(\xi), \xi)] \right\}. \quad (3)$$

In order to obtain a minimal closed set of equations for correlators $\langle \varphi \rangle$, $\langle \varphi\eta \rangle$, $\langle \varphi\varphi \rangle$, we introduce the generating functional [2]:

$$Z[J] = \langle F_z(t, t') F_{\bar{z}}(\bar{t}, \bar{t}') \exp \left[\int dud\tau J(u, \tau)\eta(u, \tau) \right] \rangle \quad (4)$$

and apply to it cumulant expansion [4,5]. Differentiating it twice by J , putting then $J=0$ and assuming, that the total stochastic ensemble of fields φ and η is Gaussian, one gets the following system of equations of bilocal approximation:

$$\langle \varphi(x, t) \rangle =$$

$$= 3\lambda \int_0^t dt' \int dy (Dz)_{xy} K_z(t, t') \int_{t'}^t d\xi \phi_{cl}(z(\xi)) \langle \varphi(z(\xi), \xi)\eta(y, t') \rangle \langle F_z(t, t') \rangle, \quad (5)$$

$$\begin{aligned}
\langle \varphi(x, t) \eta(\bar{x}, \bar{t}) \rangle &= 2 \int (Dz)_{x\bar{x}} K_z(t, \bar{t}) \langle F_z(t, \bar{t}) \rangle + 9\lambda^2 \int_0^t dt' (Dz)_{xy} K_z(t, t') \times \\
&\times \int_{t'}^t d\xi \int_{t'}^t d\xi' \phi_{cl}(z(\xi)) \phi_{cl}(z(\xi')) \langle \varphi(z(\xi), \xi) \eta(y, t') \rangle \langle \varphi(z(\xi'), \xi') \eta(\bar{x}, \bar{t}) \rangle \langle F_z(t, t') \rangle, \\
&\langle \varphi(x, t) \varphi(\bar{x}, \bar{t}) \rangle =
\end{aligned} \tag{6}$$

$$\begin{aligned}
&= 2 \int_0^{\min(t, \bar{t})} dt' \int dy (Dz)_{xy} (D\bar{z})_{\bar{x}\bar{y}} K_z(t, t') K_{\bar{z}}(\bar{t}, t') \langle F_z(t, t') F_{\bar{z}}(\bar{t}, t') \rangle + \\
&\quad + 9\lambda^2 \int_0^t dt' \int_0^{\bar{t}} d\bar{t}' \int dy d\bar{y} (Dz)_{xy} (D\bar{z})_{\bar{x}\bar{y}} K_z(t, t') K_{\bar{z}}(\bar{t}, \bar{t}') \times \\
&\quad \times \left[\int_{t'}^t d\xi \phi_{cl}(z(\xi)) \langle \varphi(z(\xi), \xi) \eta(y, t') \rangle + \int_{\bar{t}'}^{\bar{t}} d\xi' \phi_{cl}(\bar{z}(\xi')) \langle \varphi(\bar{z}(\xi'), \xi') \eta(y, t') \rangle \right] \times \\
&\quad \times \left[\int_{t'}^t d\xi \phi_{cl}(z(\xi)) \langle \varphi(z(\xi), \xi) \eta(\bar{y}, \bar{t}') \rangle + \int_{\bar{t}'}^{\bar{t}} d\xi' \phi_{cl}(\bar{z}(\xi')) \langle \varphi(\bar{z}(\xi'), \xi') \eta(\bar{y}, \bar{t}') \rangle \right] \times \\
&\quad \times \langle F_z(t, t') F_{\bar{z}}(\bar{t}, \bar{t}') \rangle,
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
\langle F_z(t, t') \rangle &= \exp \left\{ \lambda \int_{t'}^t d\xi \left[3\phi_{cl}(z(\xi)) \langle \varphi(z(\xi), \xi) \rangle + \langle \varphi^2(z(\xi), \xi) \rangle + \right. \right. \\
&\quad \left. \left. + \frac{9\lambda}{2} \int_{t'}^t d\xi' \phi_{cl}(z(\xi)) \phi_{cl}(z(\xi')) \times \right. \right. \\
&\quad \left. \left. \times \left(\langle \varphi(z(\xi), \xi) \varphi(z(\xi'), \xi') \rangle - \langle \varphi(z(\xi), \xi) \rangle \langle \varphi(z(\xi'), \xi') \rangle \right) \right] \right\}, \tag{8} \\
\langle F_z(t, t') F_{\bar{z}}(\bar{t}, \bar{t}') \rangle &= \exp \left\{ \lambda \int_{t'}^t d\xi \left[3\phi_{cl}(z(\xi)) \langle \varphi(z(\xi), \xi) \rangle + \langle \varphi^2(z(\xi), \xi) \rangle \right] + \right. \\
&\quad \left. + \lambda \int_{\bar{t}'}^{\bar{t}} d\xi' \left[3\phi_{cl}(\bar{z}(\xi')) \langle \varphi(\bar{z}(\xi'), \xi') \rangle + \langle \varphi^2(\bar{z}(\xi'), \xi') \rangle \right] + \right. \\
&\quad \left. + \frac{9\lambda^2}{2} \left[\int_{t'}^t d\xi \int_{t'}^t d\xi' \phi_{cl}(z(\xi)) \phi_{cl}(z(\xi')) \langle \varphi(z(\xi), \xi) \varphi(z(\xi'), \xi') \rangle + \right. \right. \\
&\quad \left. \left. + \int_{\bar{t}'}^{\bar{t}} d\bar{\xi} \int_{\bar{t}'}^{\bar{t}} d\bar{\xi}' \phi_{cl}(\bar{z}(\bar{\xi})) \phi_{cl}(\bar{z}(\bar{\xi}')) \langle \varphi(\bar{z}(\bar{\xi}), \bar{\xi}) \varphi(\bar{z}(\bar{\xi}'), \bar{\xi}') \rangle + \right. \right.
\end{aligned}$$

$$\begin{aligned}
& +2 \int_{t'}^t d\xi \int_{\bar{t}'}^{\bar{t}} d\bar{\xi} \phi_{cl}(z(\xi)) \phi_{cl}(\bar{z}(\bar{\xi})) < \varphi(z(\xi), \xi) \varphi(\bar{z}(\bar{\xi}), \bar{\xi}) > - \\
& - \left(\int_{t'}^t d\xi \phi_{cl}(z(\xi)) < \varphi(z(\xi), \xi) > + \int_{\bar{t}'}^{\bar{t}} d\bar{\xi} \phi_{cl}(\bar{z}(\bar{\xi})) < \varphi(\bar{z}(\bar{\xi}), \bar{\xi}) > \right)^2 \Bigg\}. \quad (9)
\end{aligned}$$

Note, that in the stochastic quantization method the order of quantum correction of the grand ensemble of fields φ and η is determined through the maximal number of the Gaussian noise fields η in correlators. Therefore, bilocal approximation describes the first semiclassical approximation to the classical lipatonic vacuum.

Solving this system of equations and taking the asymptotics of the equal-Langevin-time solutions, when Langevin time tends to infinity, one obtains physical correlators $\langle \varphi \rangle_{vac}$, $\langle \varphi \varphi \rangle_{vac}$.

3. *Solutions of equations (5)–(9) in the lowest orders of perturbation theory.* Let us perform perturbative expansion of equations (5)–(9) up to the order of $\lambda^{1/2}$. Expanding each of the correlators in the powers of $\lambda^{1/2}$, one gets from (5)–(9):

$$\langle \varphi(x, t) \rangle^{(0)} = \langle \varphi(x, t) \eta(\bar{x}, \bar{t}) \rangle^{(1)} = \langle \varphi(x, t) \varphi(\bar{x}, \bar{t}) \rangle^{(1)} = 0, \quad (10)$$

$$\langle \varphi(x, t) \eta(\bar{x}, \bar{t}) \rangle^{(0)} = 2 \int (Dz)_{x\bar{x}} K_z(t, \bar{t}), \quad (11)$$

$$\langle \varphi(x, t) \varphi(\bar{x}, \bar{t}) \rangle^{(0)} = 2 \int_0^{\min(t, \bar{t})} dt' \int dy (Dz)_{xy} (D\bar{z})_{\bar{x}y} K_z(t, t') K_{\bar{z}}(\bar{t}, t'), \quad (12)$$

$$\langle \varphi(x, t) \rangle^{(1)} = 12\sqrt{2} \int_0^t dt' \int_{\bar{t}'}^{\bar{t}} d\tau \int dy (Dz)_{xy} K_z(t, t') \frac{\rho}{z^2(\tau) + \rho^2} \int (Dz')_{z(\tau)y} K_{z'}(\tau, t'). \quad (13)$$

In order to compute the integrals in (11)–(13), we make an assumption, that the lipaton size ρ is large enough. In the case of equation (11), one, taking into account, that the integral in the right hand side of this equation is saturated near the classical trajectory, may write this condition in the form $\rho \gg |x| + |\bar{x}|$, so that all the terms of the order of $\underline{O}(\frac{z^2(\xi)}{\rho^2})$ may be neglected in comparison with 1. Then equation (11) yields in the physical limit, when $T \equiv t - \bar{t}$ tends to infinity:

$$\frac{\omega^2}{2\pi^2} \exp \left[(\sqrt{3} - 2)\omega T - \frac{\omega}{4}(x^2 + \bar{x}^2) \right], \quad (14)$$

where $\omega \equiv 8\sqrt{3}/\rho^2$.

Note, that this correlator is a purely lipatonic effect, since it tends to zero, when the lipaton infinitely grows, so that its influence becomes negligible.

In the case of the integrals in the right hand side of equation (12) the approximation of a large lipaton means, that we cut the region of integration over y by the following conditions:

$$|y| \ll \rho - |x|, \quad |y| \ll \rho - |\bar{x}|. \quad (15)$$

We shall show below, that the contribution to $\langle \varphi(x, t) \varphi(\bar{x}, t) \rangle^{(0)}$ from the integration over all the other values of y in the case, when

$$|x| + |\bar{x}| \ll \rho, \quad (16)$$

is much smaller than the contribution from the integration over the region (15), which is equal to

$$\begin{aligned} & \langle \varphi(x, t) \varphi(\bar{x}, t) \rangle^{(0)} = \\ & = \frac{\omega^2}{8\pi^2} \int_0^t \frac{d\tau}{\text{sh}^2(2\omega\tau)} \exp \left\{ \frac{\omega}{4\text{sh}(2\omega\tau)} \left[2x\bar{x} - (x^2 + \bar{x}^2) \text{ch}(2\omega\tau) \right] + 2\sqrt{3} \omega\tau \right\}. \end{aligned} \quad (17)$$

It is easy to obtain from (17), that, when the lipaton is infinitely large, i.e. when $\omega \rightarrow 0$,

$$\langle \varphi(x, t) \varphi(\bar{x}, t) \rangle^{(0)} \rightarrow \frac{1}{4\pi^2(x - \bar{x})^2} e^{-\frac{(x - \bar{x})^2}{4t}},$$

that in the limit $t \rightarrow +\infty$ yields an ordinary free boson propagator. This fact indirectly indicates, that the contribution of the region (15) is dominant.

In the case (16), splitting the interval of integration into two parts, $0 \leq \tau \leq \frac{1}{2\omega}$ and $\frac{1}{2\omega} \leq \tau \leq t$, and using in every interval the corresponding asymptotics of the integrand, one gets

$$\begin{aligned} \langle \varphi(x, t) \varphi(\bar{x}, t) \rangle^{(0)} = & -\frac{3^{1/4}i}{8\pi} \frac{\sqrt{\omega}}{|x - \bar{x}|} H_1^{(2)}(3^{1/4}\sqrt{\omega} |x - \bar{x}|) + \\ & + \frac{2 + \sqrt{3}}{4\pi^2} \omega e^{-\frac{\omega}{4}(x^2 + \bar{x}^2)} (e^{\sqrt{3}-2} - e^{2(\sqrt{3}-2)\omega t}), \end{aligned} \quad (18)$$

where $H_1^{(2)}$ is the Hankel type-2 function. Taking the asymptotics of (18) due to (16), we have

$$\lim_{t \rightarrow +\infty} \langle \varphi(x, t) \varphi(\bar{x}, t) \rangle^{(0)} = \frac{1}{4\pi^2(x - \bar{x})^2} + \frac{(2 + \sqrt{3})e^{\sqrt{3}-2}}{4\pi^2} \omega. \quad (19)$$

The contribution to $\langle \varphi(x, t) \varphi(\bar{x}, t) \rangle^{(0)}$ from the integration over those values of y , which do not satisfy the condition (15) (i.e. from the region, where the influence of the lipaton is negligible, and the motion is approximately free), is equal to $\frac{\omega}{64\sqrt{3}\pi^2}$. It is of the same order as the second term in the right hand side of (19) and, due to (16), is much smaller than the first term. Hence, the first correction to the free propagator of quantum fluctuation is a constant, which is equal to

$$\frac{\omega}{4\pi^2} \left((2 + \sqrt{3})e^{\sqrt{3}-2} + \frac{1}{16\sqrt{3}} \right). \quad (20)$$

Finally, using the approximation of the large lipaton in the region $|y| \ll \rho - |x|$, one gets from equation (13) in the limit $t \rightarrow +\infty$:

$$\langle \varphi(x, t) \rangle^{(1)} = \frac{3^{3/4}\omega^{5/2}e^{(\sqrt{3}-2)\omega t - \frac{\omega^2 t^2}{4}}}{32\pi^2} \int_0^t di' \int_{t'}^t d\tau a^2 \left\{ 1 + \frac{a-1}{\sqrt{3}[1 + 2\text{cth}(\omega(\tau - t'))]} \right\}, \quad (21)$$

where

$$a \equiv \left\{ 1 - \frac{1}{4} \text{sh}(2\omega(\tau - t')) \left[1 + 2\text{cth}(\omega(\tau - t')) \right] \right\}^{-1}.$$

We see, that $\lim_{\omega \rightarrow 0} \langle \varphi(x, t) \rangle^{(1)} = 0$, because, when the influence of the lipaton is negligible, there can not exist any corrections to $\langle \varphi(x, t) \rangle^{(0)}$ of the order of $\lambda^{1/2}$.

Note, that the integral in the right hand side of (21) diverges at $\tau = t'$, since we used unregularized Langevin equation. In order to avoid this divergence, one needs to smear $\delta(t - t')$ in (1), using stochastic regularization scheme, based on a non-Markovian generalization of the Parisi-Wu approach [8]. When t tends to infinity, $\langle \varphi(x, t) \rangle^{(1)}$, defined through equation (21), vanishes as $\underline{O}(e^{(\sqrt{3}-2)\omega t})$.

In analogous way one may compute the leading term of the asymptotics of $\langle \varphi(x, t) \rangle^{(1)}$ at t tending to infinity due to the region of integration $|y| \geq \rho - |x|$, where the motion is approximately free. It is equal to

$$\frac{3^{5/4}}{64\pi^2\sqrt{\omega t}} \int_0^t dt' \int_{t'}^t \frac{d\tau}{(\tau - t')^2} \quad (22)$$

and also should be regularized as explained above. In the physical limit, $t \rightarrow +\infty$, this expression tends to zero as $\underline{O}(\frac{1}{t})$.

4. Conclusion. In this letter we applied stochastic quantization to derivation of equations for vacuum correlators of quantum corrections to the classical solution (lipaton) in the massless $\lambda\phi^4$ theory in the approximation, that the grand ensemble of these quantum fluctuations and stochastic noise fields is Gaussian.

The obtained equations are nonlinear integral equations of the second type and contain functional integrals only in quantum mechanical sense as integrals over trajectories, but not the integrals over fields, and, therefore, may be easily investigated in the lowest orders of perturbation theory. This work is performed in section 3 for the case, when the lipaton size is large enough (the meaning of this approximation is clarified there). The solutions are given by the formulae (10),(14),(17)-(22), from which the main one is the formula (20), which yields the first constant correction to the free propagator of a quantum fluctuation, while the other correlators in the order of $\lambda^{1/2}$, derived from the properly regularized Langevin equation (see the end of section 3), vanish in the physical limit, $t \rightarrow +\infty$, according to the formulae (14),(21) and (22).

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