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NEAREST NEIGHBOR TWO-POINT CORRELATION FUNCTION OF THE Z-INVARIANT EIGHT-VERTEX MODEL

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The nearest neighbor two-point correlation function of the Z-invariant inhomogeneous eight-vertex model in the thermodynamic limit is computed using the free field representation.

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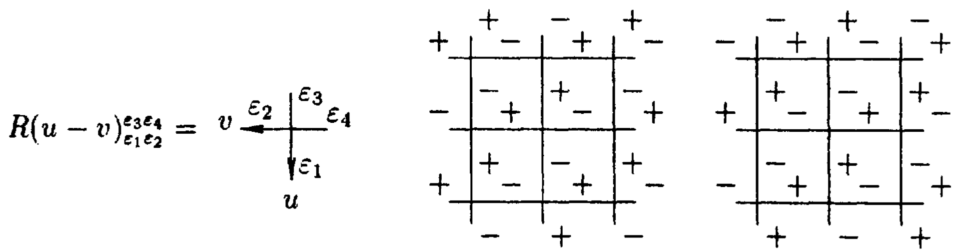
Recently, a free field construction for correlation functions of the (Z-invariant) eight-vertex model [1, 2] has been proposed [3] within the algebraic approach to integrable models of statistical mechanics [1,4–7]. The free field representation provides explicit formulas for multipoint correlation functions on the infinite lattice. However, the resulting expressions given in terms of a certain series of multiple integrals turn out to be rather cumbersome. In this letter we give an explicit expression for the nearest neighbor two-point correlation function in terms of a single two-fold integral, and perform some checks. We also discuss the independence of the integral representations of the free parameter u_0 of the vertex-face correspondence entering into the free fields construction [3].



(a) (b)

Fig.1. (a) definition of the weight matrix; (b) two degenerate ground states

Let us briefly recall the notations used in Ref. [3] (see Ref. [1] for a complete definition of the eight-vertex model). The fluctuating variables $\varepsilon = \pm 1$ are situated at edges of the square lattice. To each configuration of variables ε_1, ε_2, ε_3, ε_4 ordered around a vertex a local Boltzmann weight $R_{\varepsilon_1\varepsilon_2}^{\varepsilon_3\varepsilon_4}$ is associated, as it is shown in Fig. 1a. The nonzero

Boltzmann weights can be parametrized as follows [1]

$$\begin{aligned}
R(u)^{++}_{++} = R(u)^{--}_{--} = &\ \ a(u) = -\mathrm{i}\rho(u)\,\theta_4\left(\mathrm{i}\tfrac{\epsilon}{\pi};\mathrm{i}\tfrac{2\epsilon r}{\pi}\right)\theta_4\left(\mathrm{i}\tfrac{\epsilon}{\pi}u;\mathrm{i}\tfrac{2\epsilon r}{\pi}\right)\theta_1\left(\mathrm{i}\tfrac{\epsilon}{\pi}(1-u);\mathrm{i}\tfrac{2\epsilon r}{\pi}\right),\\
R(u)^{+-}_{+-} = R(u)^{-+}_{-+} = &\ \ b(u) = -\mathrm{i}\rho(u)\,\theta_4\left(\mathrm{i}\tfrac{\epsilon}{\pi};\mathrm{i}\tfrac{2\epsilon r}{\pi}\right)\theta_1\left(\mathrm{i}\tfrac{\epsilon}{\pi}u;\mathrm{i}\tfrac{2\epsilon r}{\pi}\right)\theta_4\left(\mathrm{i}\tfrac{\epsilon}{\pi}(1-u);\mathrm{i}\tfrac{2\epsilon r}{\pi}\right),\\
R(u)^{-+}_{+-} = R(u)^{+-}_{-+} = &\ \ c(u) = -\mathrm{i}\rho(u)\,\theta_1\left(\mathrm{i}\tfrac{\epsilon}{\pi};\mathrm{i}\tfrac{2\epsilon r}{\pi}\right)\theta_4\left(\mathrm{i}\tfrac{\epsilon}{\pi}u;\mathrm{i}\tfrac{2\epsilon r}{\pi}\right)\theta_4\left(\mathrm{i}\tfrac{\epsilon}{\pi}(1-u);\mathrm{i}\tfrac{2\epsilon r}{\pi}\right),\\
R(u)^{--}_{++} = R(u)^{++}_{--} = &-d(u) = -\mathrm{i}\rho(u)\,\theta_1\left(\mathrm{i}\tfrac{\epsilon}{\pi};\mathrm{i}\tfrac{2\epsilon r}{\pi}\right)\theta_1\left(\mathrm{i}\tfrac{\epsilon}{\pi}u;\mathrm{i}\tfrac{2\epsilon r}{\pi}\right)\theta_1\left(\mathrm{i}\tfrac{\epsilon}{\pi}(1-u);\mathrm{i}\tfrac{2\epsilon r}{\pi}\right),
\end{aligned}\tag{1}$$

where $\theta_j(u;\tau)$ is the standard jth theta function with the basic periods 1 and τ ($\mathrm{Im}\,\tau > 0$). The normalization factor $\rho(u)$ is irrelevant for correlation functions.

For definiteness, let us consider the model in the antiferroelectric phase restricting values of the parameters ϵ, r, u to be real numbers in the region $\epsilon > 0$, $r > 1$, $-1 < u < 1$. For fixed r the parameter ϵ measures deviation from the criticality. In the limit $\epsilon \to 0$ the model has a second order phase transition. In the 'low temperature' limit $\epsilon \to \infty$ the system falls into one of two ground states (Fig. 1b) indexed by $i = 0, 1$.

Let $P^{(i)}_{\varepsilon}$ be the probability in the thermodynamic limit that the spin in the 'central' edge is fixed to be ε. The label (i) indicates that spins at the edges situated 'far away' from the origin are the same as in the ith ground state so that in the low-temperature limit $\epsilon \to \infty$ the probability is nonvanishing for $\varepsilon = (-)^i$. It has been shown in Ref. [3] that the one-point correlation function

$$g^{(i)}_1 = \sum_{\varepsilon} \varepsilon P^{(i)}_{\varepsilon}$$

is recovered from the bosonization procedure. The resulting integral representation

$$g^{(i)}_1 = (-)^{i+1} 2\frac{\vartheta_1'(0)}{\vartheta_4(0)}\int_{C_0}\frac{dv}{2\pi\mathrm{i}}\frac{h_4(v)}{h_1(v)} \tag{2}$$

can be reduced to the Baxter – Kelland formula for the spontaneous staggered polarization [8]. Here we used the notations $h_j(u) = \theta_j(u/r;\mathrm{i}\pi/\epsilon r)$ and $\vartheta_j(u) = \theta_j(u;\mathrm{i}\pi/\epsilon)$. The integration contour C_0 goes over the imaginary period of the theta functions (from some complex v_0 to $v_0 + \mathrm{i}\pi/\epsilon$) so that $-1 < \mathrm{Re}\, v < 0$.

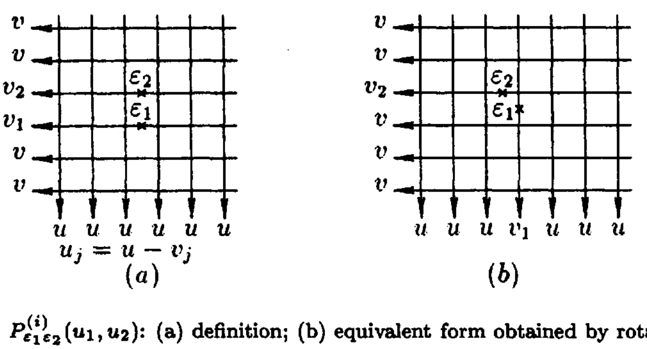


Fig.2. Probability $P^{(i)}_{\varepsilon_1\varepsilon_2}(u_1,u_2)$: (a) definition; (b) equivalent form obtained by rotating the line v_1

Consider now the inhomogeneous eight-vertex model on the lattice where the spectral parameters in two adjacent rows are u_1, u_2 respectively, while ϵ, r are the same for all sites [2]. Let $P^{(i)}_{\varepsilon_1\varepsilon_2}(u_1,u_2)$ denotes the probability of the configuration with two fixed variables as it is shown in Fig. 2a. Here (i) fixes the conditions at the infinity so that $P^{(i)}_{\varepsilon_1\varepsilon_2} \to 0$ unless $\varepsilon_1 = -\varepsilon_2 = (-)^i$ as $\epsilon \to \infty$. The main statement of this letter is that the free field construction [3] allows one to express the nearest neighbor two-point correlation function

$$g_2^{(i)}(u_1-u_2) \equiv \sum_{\varepsilon_1,\varepsilon_2=\pm} \varepsilon_1\varepsilon_2 P^{(i)}_{\varepsilon_1,\varepsilon_2}(u_1,u_2)$$

in terms of a two-fold contour integral as follows:

$$\begin{aligned} g_2^{(i)}(u) &= -2\frac{(\vartheta_1'(0))^2}{\vartheta_4(0)}\vartheta_1(u)\frac{h_4(u)}{h_1(u)}\int_{C_1}\frac{dv_1}{2\pi i}\int_{C_2}\frac{dv_2}{2\pi i}\frac{\vartheta_4(v_1+v_2-u)}{\vartheta_1(v_1-u)\vartheta_1(v_2-u)}\times \\ &\quad \times\vartheta_1(v_1-v_2)\frac{h_4(v_1-v_2+1)}{h_1(v_1-v_2+1)}\prod_{j=1}^{2}\frac{1}{\vartheta_1(v_j)}\frac{h_1(v_j)}{h_4(v_j)}. \end{aligned} \tag{3}$$

The integration contours $C_{1,2}$ go over the same imaginary period as C_0 so that $-1 < \operatorname{Re} v_1 < u < \operatorname{Re} v_2 < 1$, $\operatorname{Re} v_2 - \operatorname{Re} v_1 < 1$.

Briefly, to get this formula we applied the standard procedure of computing traces of bosonic operators over Fock spaces [7] and then proceeded as in the one-point function case (see Appendix D of Ref. [3]) by applying various identities for theta functions to provide summation over the variables of the related solid-on-solid model including infinite summation over Fock spaces. We will not go into technical details of this procedure since they are more cumbersome than instructive.

Let us only make a remark on the free parameter u_0 in the free field representation [3]. Although the explicit formulas for correlation functions in terms of traces of bosonic fields do contain u_0, it is evident from the physical grounds that correlation functions must be u_0-independent. Since the free field representation is based on some assumptions, it is important to prove this statement for the resulting integral representations for correlation functions. It can be easily checked that these expressions are nonvanishing double periodic meromorphic functions of u_0 with two generically incommensurate real periods 2 and $2r$, which proves their u_0-independence. We used the fact that $g_2^{(i)}$ is independent of u_0 to fix $u_0 = u_2 + i\pi/2\epsilon$ in Eq. (3). In addition, to make sure, we also obtained the formula for $g_2^{(i)}$ with general u_0 (which turns out to be more cumbersome) and checked numerically that it does not depend on u_0.

To support the validity of the integral representation (3) we compared it with the known results.

• The partition function differentiation method [2][1]. The probabilities $P^{(i)}_{\varepsilon_1\varepsilon_2}(u_1,u_2)$ are equal to those shown in Fig. 2b because of the Z-invariance. So the correlation function $g_2^{(i)}(u)$ can be calculated as

$$g_2^{(i)}(u) = \left(a\frac{\partial}{\partial a} - b\frac{\partial}{\partial b} - c\frac{\partial}{\partial c} + d\frac{\partial}{\partial d}\right)\log\kappa(a,b,c,d)\bigg|_{u,\epsilon,r}. \tag{4}$$

Here κ is the partition function per site as a function of the Boltzmann weights [1],

$$\log\kappa(a,b,c,d) = \log(c+d) + \sum_{m=1}^{\infty}\frac{(x^{3m}-p^{m/2})(1-p^{m/2}x^{-m})(x^m+x^{-m}-z^m-z^{-m})}{m(1-p^m)(1+x^{2m})}$$

with $x = e^{-\epsilon}$, $p = x^{2r}$, $z = e^{\epsilon(1-2u)}$, and the derivatives are taken at the point characterized by the parameters u, ϵ, r according to Eq. (1).

[1] We would like to thank Prof. J. H. H. Perk for pointing to us the possibility of this check.

We checked numerically that the results of (3) and (4) coincide at least up to the fifth decimal digit in a wide region of values of ϵ and r.

• The limiting case $r \to \infty$. In this limit the Boltzmann weights (1) become those of the six-vertex model in the antiferroelectric regime. The integral representation for correlation functions of the antiferroelectric six-vertex model is known [9, 5]. Analytically the formula (3) gives another integral representation in this limit, but numerically the results of integration according to both formulas coincide up to the sixth decimal digit.

• The limiting case $\epsilon \to 0$. This is the critical region of the eight-vertex model. Using the Baxter duality transformation for the Boltzmann weights [1] one can map the model in this region onto the six-vertex model in the gapless regime whose correlation functions were found in Ref. [10] (see also Ref. [11]). Performing the duality transform at the level of correlation functions [12] one has to compare our answer with the following correlation function

$$g_2^{JM}(\beta_1-\beta_2) \equiv -2 < E_{-+}^{(1)} E_{+-}^{(2)} > (\beta_1, \beta_2)$$

in the notations of Ref. [10] with identification $\nu = 1/r$, $\beta_j = \mathrm{i}\pi u_j$. The integral representation for this quantity can be written as follows [10]

$$\begin{aligned} g_2^{JM}(\beta) &= -2\frac{\operatorname{sh}\beta}{\operatorname{sh}\nu\beta}\int_{C_-}\frac{dv_1}{2\pi\mathrm{i}}\int_{C_+}\frac{dv_2}{2\pi\mathrm{i}}\frac{1}{\operatorname{sh}(v_1-\beta)\operatorname{sh}(v_2-\beta)}\times \\ &\quad\times\frac{\operatorname{sh}(v_1-v_2)}{\operatorname{sh}\nu(v_1-v_2+\mathrm{i}\pi)}\prod_{j=1}^{2}\frac{\operatorname{sh}\nu v_j}{\operatorname{sh} v_j}. \end{aligned} \tag{5}$$

Here the contours $C_\pm$ are from $-\infty$ to $+\infty$. They are chosen in such a way that $\beta + \pi\mathrm{i}$ (resp. β) is above (resp. below) C_+ and β (resp. $\beta - \pi\mathrm{i}$) is above (resp. below) C_-. By checking it directly one shows that the limit $\epsilon \to 0$ of Eq. (3) coincides with Eq. (5).

• The $r = 2$ case. Under such specification the eight-vertex model is equivalent to a two of non-interacting Ising models [1]. In this case $-g_2^{(i)}$ coincides with the nearest neighbor diagonal correlator of the inhomogeneous (Z-invariant) Ising model in the ferromagnetic regime [13] (see also Refs. [2, 14, 15]),

$$-g_2^{(i)}(u) = \langle \sigma_{m,n}\sigma_{m+1,n+1}\rangle = \frac{1}{\pi}\frac{\theta_4(0;\mathrm{i}\frac{2\epsilon}{\pi})}{\theta_3(0;\mathrm{i}\frac{2\epsilon}{\pi})}\frac{\theta_2'(\mathrm{i}\frac{\epsilon}{\pi}u;\mathrm{i}\frac{2\epsilon}{\pi})}{\theta_1(\mathrm{i}\frac{\epsilon}{\pi}u;\mathrm{i}\frac{2\epsilon}{\pi})}, \tag{6}$$

where $\sigma_{m,n}$ is the spin variable at the site (m, n) of the square lattice. As we show in the Appendix, Eq. (3) reduces to this formula at $r = 2$.

We hope that a similar integral representation can be obtained starting from the free field representation for other multipoint correlation functions, in particular, for two-point functions with separation of 2 lattice sites or more. It would be also very interesting to understand whether the elliptic algebra approach proposed in Refs. [16] would lead to another bosonization prescription and give a direct procedure of obtaining the integral representations of the correlation functions of the eight-vertex model.

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Appendix. Let us obtain (6) from (3) in the Ising model case $r = 2$. The integrand of (3) is antisymmetric with respect to the permutation of v_1 and v_2. This allows one to perform the second integration simply by taking the residues at the pole $v_2 = u$. Using the identity

$$\frac{\vartheta_4(u)}{\vartheta_1(u)}\frac{h_1(u)}{h_4(u)} = \frac{\vartheta_4(0)}{\vartheta_1'(0)}\frac{h_1'(0)}{h_4(0)}\frac{h_1(1)}{h_4(1)}\frac{h_4(u+1)}{h_1(u+1)} \qquad \text{(for } r = 2\text{)},$$

which is valid for $r = 2$, the resulting expression can be represented in the following form

$$g_2^{(i)}(u) = 2\frac{h_1'(0)}{h_4(0)}\frac{h_1(1)}{h_4(1)}J(u), \qquad J(u) = \int_{C_1}\frac{dv}{2\pi\mathrm{i}}\frac{h_4(v+1)h_4(v-u+1)}{h_1(v+1)h_1(v-u+1)}. \tag{A.1}$$

The function $J(u)$ satisfies the following determining properties

$$J\left(u+\frac{\mathrm{i}\pi}{\epsilon}\right) = J(u), \qquad J(-u) = J(u), \qquad J\left(\frac{\mathrm{i}\pi}{2\epsilon}\right) = \frac{1}{2\epsilon},$$
$$J(u+2) = -J(u) + \frac{h_4(u)h_4(0)}{h_1(u)h_1(0)},$$

$J(u)$ is regular on the strip $-2 < \mathrm{Re}\, u < 2$, which fix it completely to be

$$J(u) = -\frac{1}{2\epsilon}\frac{\theta_2(0,\mathrm{i}\frac{2\epsilon}{\pi})}{\theta_1'(0,\mathrm{i}\frac{2\epsilon}{\pi})}\frac{\theta_2'(\mathrm{i}\frac{\epsilon}{\pi}u;\mathrm{i}\frac{2\epsilon}{\pi})}{\theta_1(\mathrm{i}\frac{\epsilon}{\pi}u;\mathrm{i}\frac{2\epsilon}{\pi})}.$$

Passing to the conjugate module in the coefficient at $J(u)$ in Eq. (A.1) one gets (6).

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