РОССИЙСКАЯ АКАДЕМИЯ НАУК

ПИСЬМА
В ЖУРНАЛ ЭКСПЕРИМЕНТАЛЬНОЙ
И ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

ОСНОВАН В 1965 ГОДУ
ВЫХОДИТ 24 РАЗА В ГОД

ТОМ 68, ВЫПУСК 5
10 СЕНТЯБРЯ, 1998

Pis'ma v ZhETF, vol.68, iss.5, pp.353 - 358

THE CONTINUUM LIMIT AND INTEGRAL VACUUM CHARGE

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Submitted 14 July 1998
Resubmitted 6 August 1998

We consider a topologically trivial field theory defined in a large box of size L and exploit the enumeration of states to point out that the vacuum charge is integral contrary to what a commonly used formula seems to suggest. We show that the large L limit is subtle: Standard anticommutation relations require the presence of a somewhat unfamiliar normalization factor which in turn leads to observable effects for the vacuum charge density.

PACS: 03.70.+k, 11.10.St

Introduction. In this note we consider the continuum limit of a field theory defined in a large box of size L. In the limit $L \to \infty$, all discrete states apart from bound states become continuum states. According to most textbooks, quantum field theory is supposed to be well understood in this limit. We show, however, that the limiting procedure is subtle and unless proper care is exercized the usual prescriptions may give incorrect results. In particular, we show an example where the charge of the system seems to be a continuous function of the background field whereas it should be integer-valued.

An example which we have discussed in previous papers (which are referred to as CDI [1] and CD2 [2]) is provided by the second quantized Dirac theory in the presence of a one dimensional four-vector potential vanishing at spatial infinity. The usual limiting procedure for counting states in the continuum limit is given by

$$\sum(states) \to \frac{1}{\pi}\int^{\infty} dk \left(L + \frac{d\delta}{dk}\right) \tag{1}$$

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where $\delta(k)$ is the scattering phase shift for a particle of momentum k in that limit. Yet as we (and many others) have argued the vacuum charge Q_0 of the system defined in the normal way by the spectral asymmetry

$$Q_0 = \frac{1}{2}\left\{\sum_k(\text{states with } E>0) - \sum_k(\text{states with } E<0)\right\} \tag{2}$$

which is obviously integer-valued in a box of finite size L is given in the continuum limit using Eq. (1) by

$$Q_0 = \frac{1}{2}\left\{\frac{1}{\pi}\left(\delta_+(\infty) - \delta_+(0) - \delta_-(\infty) + \delta_-(0)\right) + N_+ - N_-\right\} \tag{3}$$

where $+$ and $-$ refer to electron and positron scattering phase shifts and N_+ and N_- are the number of positive and negative energy bound states. Since in one-dimension (see CDI)

$$\delta_\pm(\infty) = \pm\int_{-\infty}^{\infty} V(z)dz$$

this implies for the potential $V(z) = \lambda\delta(z)$ that $Q_0 = \lambda/\pi$ which is a continuous function of λ. The correct result is calculated using Eq. (2) in CD2 to be

$$Q_0 = Int\left[\frac{\lambda}{\pi} + \frac{1}{2}\right] \tag{4}$$

where *Int* denotes integral part of, which is obviously an integer. The erroneous result of Eq. (3) for the vacuum charge for this model is found in many places besides CDI, for example in Refs. [3] and [4]. The same error may also be responsible for similar results where other quantised quantities such as baryon charge or angular momentum seem to be given non-quantised values [5, 6] for no apparent reason. It is interesting that in the same context an explicit counting argument gives quantized values as expected [6].

In CDI following Barton [7] we counted states and showed that the number of both positive and negative energy states is unchanged when a potential is switched on from zero Thus the vacuum charge is still zero in the presence of a small potential: there is a bound state but the number of continuum states has decreased by one. As the potential increases in strength Q_0 changes by one according to Eq. (2) whenever a state crosses $E = 0$. So why is the erroneous result obtained and how can it be avoided? We now turn to these points.

In order to quantize the model consistently with standard anticommutation relations it is essential that eigenfunctions be normalised to unity. We shall see that to ensure correct normalisation we have to include an unfamiliar normalisation factor N which reduces to unity as $L \to \infty$; in fact $N^2 = 1 + \tilde{O}(1/L)$. It turns out that the expression for N^2 involves the phase shifts in a way reminiscent of the incorrect result of Eq. (3). We obtain a new expression for the charge density differing from the conventional one by terms of order $\sim 1/L$. We find a finite change in the vacuum charge (defined as the space integral of the vacuum charge density) induced by the spectral asymmetry residing in a certain region of space of the system. It is crucial to get this $1/L$ behaviour correct if we wish to sum over states and convert sums to integrals via

$$\sum_k \to \frac{L}{\pi}\int_0^\infty dk \tag{5}$$

There are topologically nontrivial models where the Dirac particle is coupled to a soliton; the attending zero modes induce non-integer values for the vacuum charge [8]. However in problems with trivial topology such as ours Q_0 is integral.

1. The Normal Modes. The potential $V(z)$ is symmetric and is taken to vanish for $|z| > a$. We take the system in a box of length $2L$ with periodic boundary conditions $\psi(-L) = \psi(L)$. Wavefunctions are classified according to the magnitude of the wavevector outside the well and parity. Positive energy solutions outside the potential take the form

$$u_{e,k}(z) = \frac{N_{e+}(k)}{\sqrt{L}}\sqrt{\frac{E+m}{2E}}\begin{pmatrix} \cos(kz \pm \Delta_{e+}) \\ 0 \\ \dfrac{ik}{E+m}\sin(kz \pm \Delta_{e+}) \\ 0 \end{pmatrix} \tag{6}$$

$$u_{o,k}(z) = \frac{N_{o+}(k)}{\sqrt{L}}\sqrt{\frac{E+m}{2E}}\begin{pmatrix} i\sin(kz \pm \Delta_{o+}) \\ 0 \\ \dfrac{k}{E+m}\cos(kz \pm \Delta_{o+}) \\ 0 \end{pmatrix} \tag{7}$$

The subscript + in the phase shifts Δ refers to the energy sign. Similar expressions are valid for negative energy states $v_{e,k}$, $v_{o,k}$ provided we replace E by $|E|$ and change notation from Δ_{e+}, Δ_{o+}, $N_{e+}(k)$, $N_{o+}(k)$ to Δ_{e-}, Δ_{o-}, $N_{e-}(k)$, $N_{o-}(k)$. We also quote for future reference the form of the even bound state wavefunction outside the well

$$u_b(|z| > a) = C\begin{pmatrix} 1 \\ 0 \\ i\dfrac{m-E_b}{\kappa} \\ 0 \end{pmatrix} e^{-\kappa z}, \kappa = \sqrt{m^2 - E_b^2} \tag{8}$$

We insist on the normalization

$$\int_{-L}^{L} dz \psi_k^\dagger(z)\psi_k(z) = 1 \tag{9}$$

for all eigenstates of the Hamiltonian. For the bound state the appropriate value of C to ensure correct normalization depends on the detailed behaviour of the potential.

The Landau-Lifshitz-Stone lemma. We rederive a result originally due to Stone [9] which itself is based on a problem in Landau and Lifshitz [10]. We start with the Dirac equation; the argument is equally valid for either positive or negative energy solutions, thus $E_k = \pm\sqrt{k^2 + m^2}$

$$\frac{1}{i}\alpha_z \frac{du_{k'}}{dz} + m\beta u_{k'} = E_{k'}u_{k'} + V(z)u_{k'} \tag{10}$$

and left-multiply by $u_k^\dagger$

$$\frac{1}{i}u_k^\dagger \alpha_z \frac{du_{k'}}{dz} + mu_k^\dagger \beta u_{k'} = E_{k'}u_k^\dagger u_{k'} + V(z)u_k^\dagger u_{k'} \tag{11}$$

Write the Dirac equation for u_k, take the Hermitian conjugate and right-multiply by $u_{k'}$

$$-\frac{1}{i}\frac{du_k^\dagger}{dz}\alpha_z u_{k'} + mu_k^\dagger \beta u_{k'} = E_k u_k^\dagger u_{k'} + V(z)u_k^\dagger u_{k'} \tag{12}$$

Subtract (12) from (11) to get

$$\frac{1}{i}\frac{d}{dz}\left(u_k^\dagger \alpha_z u_{k'}\right) = (E_{k'} - E_k)\, u_k^\dagger u_{k'} \tag{13}$$

Integrating over z from $z_1$to z_2:

$$\frac{1}{i}\left[u_k^\dagger \alpha_z u_{k'}\right]_{z_1}^{z_2} = (E_{k'} - E_k)\int_{z_1}^{z_2} u_k^\dagger u_{k'}\, dz \tag{14}$$

Take $k' = k + dk$ in the above equation and divide by dk:

$$\frac{1}{i}\left[u_k^\dagger \alpha_z \frac{du_k}{dk}\right]_{z_1}^{z_2} = \frac{dE}{dk}\int_{z_1}^{z_2} u_k^\dagger u_k dz \quad \text{or} \quad \frac{1}{i}\left[u_k^\dagger \alpha_z \frac{du_k}{dk}\right]_{z_1}^{z_2} = \frac{k}{E}\int_{z_1}^{z_2} u_k^\dagger u_k dz \tag{15}$$

This is the key equation. Its power lies in the fact that to evaluate the left hand side for $|z_1|, |z_2| > a$ it suffices to use the asymptotic expressions Eqs. (6, 7) for the wavefunctions where only the phase shifts appear.

Normalization of eigenfunctions. Note that the normalisation condition (9) together with periodic boundary conditions entail restrictions on k. Apply relation (15) at the endpoints $z_1 = -L, z_2 = L$. Then in the left hand side we only need the asymptotic expressions Eqs. (6, 7) and on the right hand side we can use Eq.(9) to set the integral equal to unity. The evaluation of the left hand side simplifies because of the periodic boundary conditions. We thus obtain

$$N_{e,o\pm}(k) = \frac{1}{\sqrt{1 + \frac{1}{L}\frac{d\Delta_{e,o\pm}}{dk}}} \tag{16}$$

and therefore

$$N_{e,o\pm}^2(k) = \frac{1}{1 + \frac{1}{L}\frac{d\Delta_{e,o\pm}}{dk}} \simeq 1 - \frac{1}{L}\frac{d\Delta_{e,o\pm}}{dk} \tag{17}$$

for L large. Note that the quantities $N_{e,o\pm}^2(k) - 1$ vanish both when $V = 0$ and in the limit $L \to \infty$. Eq. (16) is the main result of this note.

2. The Vacuum Charge Density. We focus on the charge density

$$\rho_k(z) = \psi_k^\dagger(z)\psi_k(z) \tag{18}$$

of an eigenstate $\psi_k(z)$ (of definite parity and sign of energy) corresponding to a particular wavevector k. $\rho_k(z)$ can be written down for the scattering states for $|z| > a$ by using the asymptotic forms (6) and (7) of the wavefunctions. The charge density outside the well in the case of a positive energy even parity wavefunction is given by

$$\rho_{e+}(k, z) = \frac{1}{2L}\left(1 - \frac{1}{L}\frac{d\Delta_{e+}}{dk}\right)\cdot\left(1 + \frac{m}{E}\cos(2kz + \Delta_{e+}(k))\right) \tag{19}$$

(even parity – negative energy and odd parity wavefunctions give similar expressions). Eq.(19) consists of the expected $1/2L$ part, an oscillating part and a constant background $-(1/2L^2)d\Delta_{e+}/dk$. The latter is purely a consequence of the somewhat unfamiliar normalization factor. Integration over k via Eq.(5) and subtraction of the analogous contribution from negative energy states leads to an observable distortion of the vacuum charge density.

The charge $Q_{k,ext}$ outside the well due to this state

$$Q_{k,ext} = \left(\int_{-\infty}^{-a} + \int_{a}^{\infty}\right) dz \rho_k(z) = 2\int_{a}^{\infty} dz \rho_k(z) \tag{20}$$

since V is symmetric. To order $1/L$

$$Q_{k,ext,even,\pm} = 1 - \frac{a}{L} - \frac{m}{2LEk}\sin 2\left(ka + \Delta_{e\pm}(k)\right) - \frac{1}{L}\frac{d\Delta_{e\pm}}{dk} \tag{21}$$

$$Q_{k,ext,odd,\pm} = 1 - \frac{a}{L} + \frac{m}{2LEk}\sin 2\left(ka + \Delta_{o\pm}(k)\right) - \frac{1}{L}\frac{d\Delta_{o\pm}}{dk} \tag{22}$$

The charge of the bound state outside the well can be calculated from Eq. (8)

$$Q_{b,ext} = C^2 \frac{2m}{\kappa(E+m)} e^{-2\kappa a} \tag{23}$$

In the absence of the potential the first two terms in Eqs. (21, 22) would still be there. We wish to calculate $Q_{0,ext}$ defined as the part of Q_0 residing outside the well. (Since Q_0 itself vanishes this charge is cancelled exactly by an opposite charge residing inside the well.) We see that the continuum contribution to the vacuum charge outside the well resulting from the last term in (21, 22) is given by

$$Q_{0,ext} = \frac{1}{2\pi}\left(\delta_+(\infty) - \delta_+(0) - \delta_-(\infty) + \delta_-(0)\right) \tag{24}$$

(where the phase shifts δ refer as before[1] to the sum of the even and odd phase shifts), an expression which is very similar to Eq. (3) and which will in general give non-integral values for $Q_{0,ext}$. To get the total continuum contribution we should integrate over the terms that depend on the mass explicitly. The final result is in general non-integral since there is no reason why the charge inside or outside a particular region of space should be integer-valued.

Conclusion. We investigate a commonly used formula which seems to give non-integral vacuum charge in the continuum limit. Enumeration of states establishes a one-to-one correspondence between states for various values of the potential (including $V = 0$) thus ensuring that the vacuum charge is integer valued. We show that consistency with standard anticommutation relations requires the presence of a somewhat unfamiliar normalization factor which modifies the charge density pertaining to a particular wavevector by a position-independent addend of order $1/L^2$. This term is sensitive to the violation of charge conjugation by the potential and leads to observable effects upon summation over all wavevectors. This is relevant to the calculation of the vacuum charge residing in a certain region of space. The existence of this localized charge is due to the distortion of the Dirac sea induced by the potential.

Acknowledgements. We would like to thank Gabriel Barton, Fred Goldhaber and Mike Birse for their help and encouragement. One of us (AC) wishes to thank Professor Volovik and the Low Temperature Laboratory of Helsinki University of Technology for their hospitality and EU Training and Mobility of Researches Programme Contract # ERBFMGECT980122 for its support.

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