РОССИЙСКАЯ АКАДЕМИЯ НАУК

ПИСЬМА
В ЖУРНАЛ ЭКСПЕРИМЕНТАЛЬНОЙ
И ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

ОСНОВАН В 1965 ГОДУ
ВЫХОДИТ 24 РАЗА В ГОД

ТОМ 68, ВЫПУСК 10
25 НОЯБРЯ, 1998

Pis'ma v ZhETF, vol.68, iss.10, pp.721 - 726

HOW TO DETERMINE AN EFFECTIVE POTENTIAL FOR A VARIABLE COSMOLOGICAL TERM

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Submitted 23 October 1998

It is shown that if a variable cosmological term in the present Universe is described by a scalar field with minimal coupling to gravity and with some phenomenological self-interaction potential $V(\varphi)$, then this potential can be unambiguously determined from the following observational data: either from the behaviour of density perturbations in dustlike matter component as a function of redshift (given the Hubble constant additionally), or from the luminosity distance as a function of redshift (given the present density of dustlike matter in terms of the critical one).

PACS: 98.80.Es, 98.80.Hw

It has been known for many years that the flat Friedmann – Robertson – Walker (FRW) cosmological model with cold dark matter (CDM), a positive cosmological constant $\Lambda > 0$ ($\Omega_0 + \Omega_\Lambda = 1$) and an approximately flat (or, Harrison – Zeldovich-like, $n_S \approx 1$) spectrum of primordial scalar (adiabatic) perturbations fits observational data better and has a larger admissible region of the parameters (H_0, Ω_0) than any other cosmological model with both inflationary and non-inflationary initial conditions (see, e.g., [1, 2]). Here H_0 is the Hubble constant, $\Omega_0 = 8\pi G\rho_m/3H_0^2$ includes baryons and (mainly) non-baryonic dark matter, $\Omega_\Lambda \equiv \Lambda/3H_0^2$ and the light velocity $c = 1$. This conclusion was based on the following arguments: a) relation between H_0 and the age of the Universe t_0, b) the fact that observed mass/luminosity ratio never leads to values more than $\Omega_0 \sim 0.4$ up to supercluster scales, c) comparison of cosmic microwave background (CMB) temperature anisotropies, power spectra of density and velocity matter perturbations, present abundance of galaxy clusters with predictions of cosmological models with inflationary initial conditions; d) observed values of ρ_b/ρ_m in rich galaxy clusters confronted with the range for the present baryon density ρ_b admitted by the theory of primordial (Big Bang) nucleosynthesis. I don't include gravitational lensing tests (e.g., a number of lensed quasars) here, since conclusions based on them are less definite at present; however, the most recent reconsideration [3] has also led to a low value of $\Omega_0 \sim 0.3$.

During last year two new pieces of strong evidence for $\Omega_0 < 1$ have appeared. The first (historically) of them is based on the evolution of abundance of rich galaxy clusters with redshift z [4], see also the more recent paper [5] where the value $\Omega_0 \approx 0.5 \pm 0.2$ (1σ uncertainty) is presented. Still, it should be noted that there have been already appeared some doubts on validity of the conclusion that $\Omega_0 = 1$ is really excluded [6]. Much better observational data expected in near future will help to resolve this dilemma unambiguously. The second, completely independent argument for $\Omega_0 = 0.2 - 0.4$ follows from direct observations of supernovae type Ia (SNIa) explosions at high redshifts up to $z \sim 1$ [7]. On the other hand, no direct evidence for a negative spatial curvature of the Universe (i.e., for the open FRW model) has been found. Just the opposite, the latest CMB constraints (based mainly on the results of the Saskatoon and CAT experiments) [8], galaxy abundance at high redshifts [9] and the most recent analysis of the SNIa data in terms of an effective equation of state of a component adding Ω_0 to unity [10] strongly disfavor the open CDM model without a positive cosmological constant. Of course, the possibility to have *both* a positive cosmological constant and spatial curvature of any sign is not yet excluded, but, according to the "Okkam's razor" principle, it would be desirable not to introduce one more basic novel feature of the Universe (spatial curvature) without conclusive observational evidence. In any case, in spite of many theoretical and experimental attempts to exorcize it, a Λ-term is back again.

It is clear that the introduction of a cosmological constant requires new and completely unknown physics in the region of ultra-low energies. Solutions with a cosmological constant occur in such fundamental theories as supergravity and M-theory. However, this cosmological constant is always negative and very large. As compared to such a basic "vacuum" state, a very small and positive cosmological constant allowed in the present Universe may be thought as corresponding to the energy density ε_Λ of a highly excited (though still very symmetric) "background" state. So, it need not be very "fundamental". But then it is natural to omit the assumption that it should be exactly constant. In this case the name "a cosmological term" (or a Λ-term) is more relevant for it, so I shall use this one below. The principal difference between two kinds of non-baryonic dark matter – dustlike CDM and a Λ-term – is that the latter one is not gravitationally clustered up to scales $\sim 30\ h^{-1}$ or more (otherwise we would return to the problem why Ω_0 observed from gravitational clustering is not equal to unity). Here $h = H_0/100\ \mathrm{km\,s^{-1}Mpc^{-1}}$.

On the other hand, there exists a well-known strong argument showing that a Λ-term cannot change with time as fast as the matter density ρ_m and the Ricci tensor (i.e., $\propto t^{-2}$) during the matter dominated stage (for redshifs $z < 4\cdot 10^4\ h^2$). Really, if $\varepsilon_\Lambda \propto \rho_m$, so that $\Omega_\Lambda = \mathrm{const}$, then matter density perturbations in the CDM+baryon component grow as

$$\delta \equiv \left(\frac{\delta\rho}{\rho}\right)_m \propto t^\alpha \propto (1+z)^{-3\alpha/2},\ \alpha = \frac{\sqrt{25 - 24\Omega_\Lambda} - 1}{6}.$$

As a consequence, the total growth of perturbations Δ since the time of equality of matter and radiation energy densities up to the present moment is less than in the absence of the Λ-term. If $\Omega_\Lambda \ll 1$, then $\Delta(\Omega_\Lambda) = \Delta(0)(1 - (6.4 + 2\ln h)\Omega_\Lambda)$. Since parameters of viable cosmological models are so tightly constrained that Δ may not be reduced by more than twice approximately, this type of a Λ-term cannot account for more than ~ 0.1 of the critical energy density (see [11] for detailed investigation confirming this conclusion). This, unfortunately, prevents us from natural explanation of the present Λ-term with $\Omega_\Lambda = (0.5 - 0.8)$ using "compensation" mechanisms [12] or exponential potentials with sufficiently large exponents [13]; in other words, a Λ-term cannot be produced by an exactly "tracker" field as was recently proposed in [14].

A natural and simple description of a variable Λ-term is just that which was so successively used to construct the simplest versions of the inflationary scenario, namely, a scalar field with some interaction potential $V(\varphi)$ minimally coupled to the Einstein gravity. Such an approach, though phenomenological, is nevertheless more consistent and fundamental than a commonly used attempt to decribe a Λ-term by a barotropic ideal fluid with some equation of state. The latter approach cannot be made internally consistent in case of negative pressure which is implied by observations [10], in particular, it generally leads to imaginary values of the sound velocity. On the contrary, no such problems arise using the scalar field description (this scalar field is called the Λ-field below). Of course, its effective mass $|m_\varphi^2| = |d^2V/d\varphi^2|$ should be very small to avoid gravitational clustering of this field in galaxies, clusters and superclusters. To make a Λ-term slowly varying, we assume that $|m_\varphi| \sim H_0 \sim 10^{-33}$ eV, or less (though this condition may be relaxed). Models with a time-dependent Λ-term were introduced more than ten years ago [15], and different potentials $V(\varphi)$ (all inspired by inflationary models) were considered: exponential [13, 16, 11, 17], inverse power-law [18], power-law [19], cosine [20, 17].

However, it is clear that since we know essentially nothing about physics at such energies, there exists no preferred theoretical candidate for $V(\varphi)$. In this case, it is more natural to go from observations to theory, and to determine an effective phenomenological potential $V(\varphi)$ from observational data. The two new tests mentioned above are just the most suitable for this aim. Really, using the cluster abundance $n(z)$ determined from observations and assuming the Gaussian statistics of initial perturbations (the latter follows from the paradigm of one-field inflation, and it is in agreement with other observational data), it is possible to determine a *linear* density perturbation in the CDM+baryon dust-like component $\delta(z)$ for a fixed comoving scale $R \sim 8(1+z)^{-1}h^{-1}$ Mpc up to $z \sim 1$, either using the Press-Schechter approximation, or by direct numerical simulations of nonlinear gravitational instability in the expanding Universe. $\delta(z)$ can be also determined from observation of gravitational clustering (in particular, of the galaxy-galaxy correlation function) as a function of z. On the other hand, observations of SNe at different z yield the luminosity distance $D_L(z)$ through the standard astronomical expression $m = M + 5\log D_L + 25$, where m is the observed magnitude, M is the absolute magnitude and D_L is measured in Mpc.

The aim of the present letter is to show how to determine $V(\phi)$ from either $\delta(z)$ or $D_L(z)$, and to investigate what additional information is necessary for an unambiguous solution of this problem in both cases. The idea has been already announced by the author in [21, 22], now details are given.

The derivation of $V(\varphi)$ consists of two steps. First, the Hubble parameter $H \equiv \dot{a}/a = H(z)$ is determined. Here $a(t)$ is the FRW scale factor, $1+z \equiv a_0/a$, the dot means d/dt and the index 0 denotes the present value of a corresponding quantity (in particular, $H(t_0) = H(z=0) = H_0$). In the case of SNe, the first step is almost trivial since the textbook expression for D_L reads:

$$D_L(z) = a_0(\eta_0 - \eta)(1+z), \quad \eta = \int_0^t \frac{dt}{a(t)} \,. \tag{1}$$

Therefore,

$$H(z) = \frac{da}{a^2 d\eta} = -(a_0\eta')^{-1} = \left[\left(\frac{D_L(z)}{1+z}\right)'\right]^{-1} . \tag{2}$$

Here and below, a prime denotes the derivative with respect to z. Thus, $D_L(z)$ defines $H(z)$ uniquely.

More calculations are required to find $H(z)$ from $\delta(z)$. The system of background equations for the system under consideration is:

$$H^2 = \frac{8\pi G}{3}\left(\rho_m + \frac{\dot{\varphi}^2}{2} + V\right) , \quad \rho_m = \frac{3\Omega_0 H_0^2 a_0^3}{8\pi G a^3} , \tag{3}$$

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{dV}{d\varphi} = 0 , \tag{4}$$

$$\dot{H} = -4\pi G(\rho_m + \dot{\varphi}^2) . \tag{5}$$

Eq. (5) is actually the consequence of the other two equations.

We consider a perturbed FRW background which metric, in the longitudinal gauge (LG), has the form:

$$ds^2 = (1+2\Phi)dt^2 - a^2(t)(1+2\Psi)\delta_{lm}dx^l dx^m; \quad l,m = 1,2,3 . \tag{6}$$

The system of equations for scalar perturbations reads (the spatial dependence $\exp(ik_l x^l)$, $k_l k^l \equiv k^2$ is assumed):

$$\Phi = \Psi = \dot{v} , \quad \dot{\delta} = -\frac{k^2}{a^2}v + 3(\ddot{v} + H\dot{v} + \dot{H}v) , \tag{7}$$

$$\dot{\Phi} + H\Phi = 4\pi G(\rho_m v + \dot{\varphi}\delta\varphi) , \tag{8}$$

$$\left(-\frac{k^2}{a^2} + 4\pi G\dot{\varphi}^2\right)\Phi = 4\pi G(\rho_m\delta + \dot{\varphi}\dot{\delta\varphi} + 3H\dot{\varphi}\delta\varphi + \frac{dV}{d\varphi}\delta\varphi) , \tag{9}$$

$$\ddot{\delta\varphi} + 3H\dot{\delta\varphi} + \left(\frac{k^2}{a^2} + \frac{d^2V}{d\varphi^2}\right)\delta\varphi = 4\dot{\varphi}\dot{\Phi} - 2\frac{dV}{d\varphi}\Phi . \tag{10}$$

Eq. (10) is the consequence of other ones. Here v and $\delta\varphi$ are, correspondingly, a velocity potential of a dustlike matter peculiar velocity and a Λ-field perturbation in LG, and δ is a *comoving* fractional matter density perturbation (in this case, it coincides with $(\delta\rho/\rho)_m$ in the synchronous gauge). In fact, all these perturbed quantities are gauge-invariant.

Now let us take a comoving wavelength $\lambda = k/a(t)$ which is much smaller than the Hubble radius $H^{-1}(t)$ up to redshifts $z \sim 5$. This corresponds to $\lambda \ll 2000\ h^{-1}$ Mpc at present. Then, from Eq. (10),

$$\delta\varphi \approx \frac{a^2}{k^2}(4\dot{\varphi}\dot{\Phi} - 2\frac{dV}{d\varphi}\Phi), \quad |\dot{\varphi}\dot{\delta\varphi}| \sim |\frac{dV}{d\varphi}\delta\varphi| \sim \frac{a^2H^4}{Gk^2}|\Phi| \ll \rho_m|\delta| . \tag{11}$$

Therefore, the Λ-field is practically unclustered at the scale involved. Now the last of Eqs. (7) and Eq. (9) may be simplified to:

$$\dot{\delta} = -\frac{k^2}{a^2}v, \quad -\frac{k^2}{a^2}\Phi = 4\pi G\rho_m\delta . \tag{12}$$

Combining this with the first of Eqs. (7), we return to a well-known equation for δ in the absence of the Λ-field:

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\rho_m\delta = 0 \,. \tag{13}$$

It is not possible to solve this equation analytically for an arbitrary $V(\varphi)$. Remarkably, the inverse dynamical problem, i.e. the determination of $H(a)$ given $\delta(a)$, is solvable. After changing the argument in Eq. (13) from t to a ($d/dt = aH(d/da)$), we get a first order linear differential equation for $H^2(a)$:

$$a^2\frac{d\delta}{da}\frac{dH^2}{da} + 2\left(a^2\frac{d^2\delta}{da^2} + 3a\frac{d\delta}{da}\right)H^2 = \frac{3\Omega_0 H_0^2 a_0^3\delta}{a^3} \,. \tag{14}$$

The solution is:

$$H^2 = \frac{3\Omega_0 H_0^2 a_0^3}{a^6}\left(\frac{d\delta}{da}\right)^{-2}\int_0^a a\delta\frac{d\delta}{da}\,da = 3\Omega_0 H_0^2\frac{(1+z)^2}{\delta'^2}\int_z^\infty \frac{\delta|\delta'|}{1+z}\,dz \,. \tag{15}$$

Putting $z = 0$ in this expression for H, we arrive to the expression of Ω_0 through $\delta(z)$:

$$\Omega_0 = \delta'^2(0)\left(3\int_0^\infty \frac{\delta|\delta'|}{1+z}\,dz\right)^{-1} \,. \tag{16}$$

Of course, observations of gravitational clustering can hardly provide the function $\delta(z)$ for too large z (say, for $z > 5$). However, $\delta(z)$ in the integrands in Eqs. (15), (16) may be well approximated by its $\Omega_0 = 1$ behaviour (i.e., $\delta \propto (1+z)^{-1}$) already for $z > (2-3)$. If massive neutrinos are present, one should use here the expression with α written above and with Ω_Λ substituted by Ω_ν/Ω_0 (it is assumed that ρ_m includes massive neutrinos, too).

Finally, using Eq. (16), Eq. (15) can be represented in a more convenient form:

$$\frac{H^2(z)}{H^2(0)} = \frac{(1+z)^2\delta'^2(0)}{\delta'^2(z)} - 3\Omega_0\frac{(1+z)^2}{\delta'^2(z)}\int_0^z \frac{\delta|\delta'|}{1+z}\,dz \,. \tag{17}$$

Thus, $\delta(z)$ uniquely defines the ratio $H(z)/H_0$. Of course, appearance of derivatives of $\delta(z)$ in these formulas shows that sufficiently clean data are necessary, but one may expect that such data will soon appear. Let us remind also that, for $\Lambda \equiv$ const ($V(\varphi) \equiv$ const), we have

$$H^2(z) = H_0^2(1 - \Omega_0 + \Omega_0(1+z)^3), \quad q_0 \equiv -1 + \left(\frac{d\ln H}{d\ln(1+z)}\right)_{z=0} = \frac{3}{2}\Omega_0 - 1 \,, \tag{18}$$

where q_0 is the acceleration parameter.

The second step – the derivation of $V(\varphi)$ from $H(a)$ - is very simple. One has to rewrite Eqs. (3), (5) in terms of a and take their linear combinations:

$$8\pi GV(\varphi) = aH\frac{dH}{da} + 3H^2 - \frac{3}{2}\Omega_0 H_0^2\left(\frac{a_0}{a}\right)^3 \,,$$
$$4\pi Ga^2H^2\left(\frac{d\varphi}{da}\right)^2 = -aH\frac{dH}{da} - \frac{3}{2}\Omega_0 H_0^2\left(\frac{a_0}{a}\right)^3 \,, \tag{19}$$

and then exclude a from these equations.

Therefore, the model of a Λ-term considered in this paper can account for *any* observed forms of $D_L(z)$ and $\delta(z)$ which, in turn, can be transformed into a corresponding effective potential $V(\varphi)$ of the Λ-field. The only condition is that the functions $H(z)$ obtained by two these independent ways should coincide within observational errors. $D_L(z)$ uniquely determines $V(\varphi)$, if Ω_0 is given additionally (the latter is required at the second step, in Eqs. (19)). $\delta(z)$ uniquely determines $V(\varphi)$ up to the multiplier H_0^2, the latter has to be given additionally to fix an overall amplitude. Observational tests which can falsify this model do exist. In particular, a contribution to large-angle $\Delta T/T$ CMB temperature anisotropy due to the integrated (or, non-local) Sachs-Wolfe effect presents a possibility to distinguish the model from more complicated models, e.g., with non-minimal coupling of the Λ-field to gravity or to CDM. However, the latter test is not an easy one, since this contribution is rather small and partially masked by cosmic variance.

The research was partially supported by the Russian Foundation for Basic Research, Grant 96-02-17591, and by the Russian Research Project "Cosmomicrophysics".

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