Pis'ma v ZhETF, vol.68, iss.11, pp.791 - 795

ON THE THEORY OF CHIRPED OPTICAL SOLITON IN FIBER LINES WITH VARYING DISPERSION.

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Submitted 22 October 1998

We study soliton solution of a path-averaged (in the spectral domain) propagation equation governing transmission of a chirped breathing pulse in the fiber lines with dispersion compensation. We demonstrate that the averaged Hamiltonian model correctly describes features of chirped soliton observed in numerical simulations and experiments. We show that the Hamiltonian is bounded from below if the average dispersion is anomalous ($\langle d \rangle > 0$) that together with a condition $H_{sol} < 0$ indicates stability of dispersion-managed soliton in this region.

PACS: 42.65.-k

Soliton-based high-bit-rate optical data transmission [1] is a nice example how results of the fundamental research can trigger development of a novel approach to a very important practical problem. Invention of erbium-doped fiber amplifiers providing periodic amplification of lightwave signals to compensate fiber loss gave rise to investigations of different methods to overcome other sources of signal degradation in fiber lines such as chromatic dispersion and nonlinearity. In the low-power (linear) regime the detrimental effect of the chromatic dispersion (pulse broadening) can be eliminated by the dispersion compensation. The simplest dispersion compensating system consists of a transmission fiber and equalizing fiber with the opposite dispersion allowing to reduce total dispersion of the fiber span. Soliton transmission makes positive use of nonlinearity for compensating of the chromatic dispersion effects and, at first glance, does not require any dispersion management. However, it turns out that the use of varying dispersion has also many advantages in the case of the soliton transmission. For instance, dispersion management by specially designed fiber with exponential decay of dispersion [1] transforms propagating equation for soliton in real lossy fiber to the integrable [2] nonlinear Schroedinger equation (NLSE) with suppression detrimental effects of a lumped amplification. The dispersion-managed (DM) soliton is a new kind of the information carrier whose features differ significantly from that of the soliton solution of NLSE. Large variations of the dispersion (strong dispersion management) strictly modify the pulse propagation, inducing breathing-like oscillations of the pulse width during amplification period and creating nontrivial distribution of the phase in time (pulse chirp). There are two characteristic scales in the dynamics of DM pulse: the fast dynamics corresponds to rapid oscillations of the pulse width, phase and power due periodic variations of the dispersion and periodic amplification; and the slow evolution is determined by the combined effects of nonlinearity, residual dispersion and pulse chirping. Slow dynamics of the DM soliton then can be described by the propagation equation averaged over fast oscillations. Using of a stable steady-state solution of the path-averaged equation as an information carrier provides stable data transmission along the fiber line. The small parameter in the considered problem is $\epsilon = L/Z_{NL}$, with L being a compensation period and Z_{NL} a characteristic nonlinear scale (when nonlinear effects come into play). Though DM soliton has been already rather comprehensively studied [3 – 16], due of the important practical applications it is of interest to develop different alternative theoretical methods to describe properties of DM soliton. In the present Letter solving previously derived path-average equation in the frequency domain [6, 7] we observe Gaussian core and exponentially decaying (oscillating) tails of the DM soliton. The central question for the physical relevance of any solitary waves is their stability. We prove also in this Letter that in the average Hamiltonian model the Hamiltonian is bounded from below that indicates that DM soliton is stable.

Optical pulse propagation down the cascaded transmission system with varying dispersion is governed by

$$iA_z + d(z)A_{tt} + \epsilon\, c(z)\,|A|^2 A = 0. \tag{1}$$

Here we normalize the distance z by the compensation period L, $d(z)$ and $c(z)$ are periodic functions describing varying dispersion and signal power oscillations due to loss and amplification, respectively. For notations we refer to our previous papers [15] (note, however, that here we put a small parameter ϵ outside of $c(z)$, letting $c(z)$ to be of the order of one). Chromatic dispersion $d(z) = \tilde{d}(z) + \langle d \rangle$ presents a sum of a rapidly (over one compensating period) varying high local dispersion and a constant residual dispersion.

As it has first been observed in numerical simulations in [4], DM soliton can propagate even in the region $\langle d \rangle < 0$, as opposite for traditional "bright" soliton that exists only if $\langle d \rangle > 0$. This feature can be understood considering integral pulse characteristics: width T_{int}, chirp M_{int}, spectral bandwidth Ω_{int} and averaged power P_{int} defined as

$$T_{int}^2(z) = \frac{\int t^2 |A|^2 dt}{\int |A|^2 dt}, \qquad \frac{M_{int}(z)}{T_{int}(z)} = \frac{i}{4}\, \frac{\int t(AA_t^* - A^* A_t)dt}{\int t^2 |A|^2 dt},$$

$$\Omega_{int}^2(z) = \frac{\int \omega^2 |A|^2 d\omega}{\int |A|^2 d\omega}, \qquad P_{int}(z) = \frac{\int |A|^4 dt}{\int |A|^2 dt}. \tag{2}$$

It is easy to find that

$$\frac{d}{dz}(T_{int}\, M_{int}) = d(z)\, \Omega_{int}^2 - \epsilon\, \frac{c(z)}{4}\, P_{int}. \tag{3}$$

Integrating Eq.(3) over one period we get simple explanation why DM soliton can propagate at the zero and normal average dispersion. Indeed, for the true periodic T_{int} and M_{int} we have

$$\langle d(z)\, \Omega_{int}^2 \rangle = \frac{\epsilon}{4}\, \langle c(z)\, P_{int} \rangle. \tag{4}$$

It is seen from here that the requirement $\langle d \rangle > 0$ that provides existence of the traditional soliton in the NLSE is replaced by a condition $\langle d(z) \Omega_{int}^2 \rangle > 0$ and the average dispersion can be zero and positive still holding $\langle d(z)\, \Omega_{RMS}^2 \rangle > 0$. Function $d\, \Omega_{int}^2$ plays a role of the effective dispersion of the system. Thus, propagation in the region of the negative average dispersion is possible due to variation of the pulse phase (chirp) along the section. Rapid (quasi-linear) oscillations of the phase play a crucial role in the dynamics of DM pulse. The dynamics of DM soliton can be considered as a fast oscillations of the phase and slow evolution of the amplitude. Therefore it is natural to use spectral domain [6] for the average description of the pulse evolution. Following [6, 7] let us apply the following transformation from A to a slowly varying function q based on the Fourier transform

$$A(t,z) = \int_{-\infty}^{+\infty} d\omega q(\omega,z) \exp[-i\omega t - i\omega^2 R(z)], \tag{5}$$

here $dR(z)/dz = \tilde{d}(z)$, and $\langle R \rangle = 0$. The aim of this transformation is to eliminate large coefficient $\tilde{d}$ from (1). Equation on q can now be averaged directly [6, 7] as opposite to (1). Physical interpretation of the above transformation is rather transparent [6]. In the linear propagation regime the spectral bandwidth is not changed by a rapidly varying dispersion. Only a pulse phase follows rapid oscillations of the local dispersion. Therefore, effects of nonlinearity and small residual dispersion can be accounted as a slow evolution of the envelope of the quasi-linear solution. This decomposition of the fast evolution of the phase and a slow evolution of the amplitude is effectively realized by the transform (5) in the spectral domain. Without loss of generality consider now similar to [3] a lossless model ($c(z) = 1$) with a two-step dispersion map built from a piece of a fiber with the dispersion $d_1 + \langle d \rangle$ and length l_1 followed by fiber with dispersion $d_2 + \langle d \rangle$ and length $l_2 = 1 - l_1$. Here $d_1 l_1 + d_2(1 - l_1) = 0$. For convenience (this allows below to remove ϵ from the stationary equation) we also make $\langle d \rangle$ proportional to ϵ. The inclusion of periodic amplification and dispersion compensation can be handled as separate problems, provided that amplification distance is substantially different from the period of dispersion map. Note that this is a typical situation for ultra-long transoceanic optical communication systems [8]. As a result of the averaging [6] we have integro-differential equation in the spectral domain describing slow evolution of $q(\omega, z)$

$$i\frac{\partial q(z,\omega)}{\partial z} = \omega^2 \langle d \rangle q(z,\omega) - \epsilon \int d\omega_1 d\omega_2 \frac{\sin[\mu(\omega-\omega_1)(\omega-\omega_2)]}{\mu(\omega-\omega_1)(\omega-\omega_2)} q_{\omega_1} q_{\omega_2} q^*_{\omega_1+\omega_2-\omega} = \frac{\delta H}{\delta q^*} \tag{6}$$

with the Hamiltonian

$$H = \langle d \rangle \int \omega^2 |q|^2 d\omega - \frac{\epsilon}{2} \int d\omega_1 d\omega_2 d\omega_3 d\omega_4 \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) q_{\omega_1} q_{\omega_2} q^*_{\omega_3} q^*_{\omega_4} \frac{\sin[\mu \Delta\Omega/2]}{(\mu\Delta\Omega/2)}, \tag{7}$$

here $\Delta\Omega = \omega_1^2 + \omega_2^2 - \omega_3^2 - \omega_4^2$ and $\mu = d_1 l_1$. The parameter $\mu = d_1 l_1 = R(l_1)$ introduced here is nothing more, but a characteristic of the map strength. Strong dispersion management corresponds to $\mu \gg \langle d \rangle$. It is seen from Eq.(6) that if $\langle d \rangle$ is small, as well as ϵ, then q varies on the scales much larger than the compensation period. The energy integral $E = \int |q|^2 d\omega$ is the additional conserved quantity in Eq.(6). Path-averaged soliton of the form $q(\omega, z) = F(\omega) \exp(i\,\epsilon\, k z)$ realizes extremum of the H for a fixed E:

$$\delta(H + \epsilon\, k\, E) = 0. \tag{8}$$

These equation can be viewed as a nonlinear eigenvalue problem for k and $F(\omega)$. We analyze in this paper solutions of Eq.(1) being localized in ω. Solitons arise from a balance between nonlinearity, residual dispersion and pulse chirping effect. In Fig. 1 it is shown typical solution of the stationary Eq.(8). Soliton power ($|F(\omega)|^2$) is shown in the logarithmic scale for $\mu = 5$, $\langle d \rangle = 0.1\,\epsilon$ and $k = 1$ (solid line), 0.5 (dashed line) and 0.2 (short-dashed line). In the inset soliton power is shown in the usual scale. It is seen from Fig. 1 that in agreement with the results of numerical simulations [3–5, 9, 15] of the basic Eq.(1) the obtained solution has a Gaussian core in the energy containing central region and the exponentially decaying oscillating tails. Similar to the observations of [4] for original Eq.(1) localized solutions of Eq.(6) exist even in some region where $\langle d\rangle < 0$. As is was mentioned above, this region can be found from the condition on the averaged effective dispersion $\langle d\,\Omega_{int}^2\rangle > 0$. In the limit $\mu = 0$ Eq.(6) as expected is nothing more but the well-known NLS equation and the soliton solution in this case has cosh shape. Letting $2\langle d\rangle = \epsilon$ one can write solution in this NLSE limit as $q(\omega, z) = \sqrt{k}\,\mathrm{sech}[\pi\,\omega/(2\sqrt{k})]\ \exp(i\epsilon\, k\, z/2)$. Equation (6) presents a regular way to describe DM soliton and an alternative to the path-averaged propagation equation in the time domain derived in [15] using soliton expansion in the basis of the chirped Gauss-Hermite functions. Even though in general case, Eq.(6) is not easy to solve, it allows to describe all features of path-averaged DM pulse and can be useful for stability consideration. In this paper we consider stability properties of the soliton for the case $\langle d\rangle > 0$. Dependence of Hamiltonian H on the soliton energy

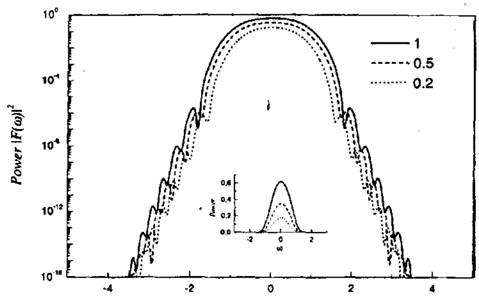

Fig.1. Shape of the soliton solution of Eq. (6) with the Gaussian central part and exponentially decaying oscillating tails is plotted in the logarithmic scale for $k = 1$ (solid line), 0.5 (dashed line) and 0.2 (short-dasged line). Here $\mu = 5$, $\langle d\rangle = 0.1\,\epsilon$. The same profiles in the usual scale are shown in the inset

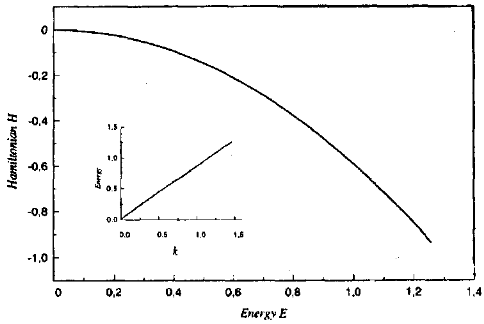

Fig.2. Dependence of the Hamiltonian H on the soliton energy E. In the inset it is shown dependence of the energy on the parameter k. As in Fig. 1 here $\mu = 5$, $\langle d\rangle = 0.1\,\epsilon$

E (for positive average dispersion) is plotted in Fig.2. Inset shows that in this range of parameters the energy dependence on the parameter k is about rectilinear. In important property of the soliton is that the Hamiltonian H is negative on the soliton solution. Next we demonstrate the thet Hamiltonian H is bounded from below for fixed energy E. Thus, if this minimum is attained on some steady-state solution this solution is stable. Effectively, we show that the Hamiltonian of Eq.(6) is bounded from below by the NLSE Hamiltonian. Substitution of simple estimates $q_{\omega_1} q_{\omega_2} q^*_{\omega_3} q^*_{\omega_4} \le |q_{\omega_1}||q_{\omega_2}||q_{\omega_3}||q_{\omega_4}|$ and $\sin(x)/x \le 1$ into the Hamiltonian H yields

$$H \ge \langle d\rangle \int \omega^2 |q|^2 d\omega - \frac{\epsilon}{2}\int d\omega_1 d\omega_2 d\omega_3 d\omega_4 \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)|q_{\omega_1}||q_{\omega_2}||q_{\omega_3}||q_{\omega_4}| =$$

$$= \langle d\rangle \int \omega^2 |q|^2 d\omega - \frac{\epsilon}{32}\int d\omega_1 d\omega_2 d\omega_3 d\omega_4 \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)(q_{\omega_1} + q^*_{\omega_1})\times$$

$$\times (q_{\omega_2} + q^*_{\omega_2})(q_{\omega_3} + q^*_{\omega_3})(q_{\omega_4} + q^*_{\omega_4}) \ge \frac{\langle d\rangle}{2\pi}\int |q_t|^2 dt - \frac{\epsilon}{16\pi}\int (|q(t)|^2 + |q(-t)|^2)^2 dt \ge$$

$$\geq \frac{\langle d \rangle}{2\pi} \int |q_t|^2 dt - \frac{\epsilon}{4\pi} \int |q(t)|^4 dt = \frac{H_{NLSE}}{2\pi} \geq -\frac{\epsilon^2}{48\,\langle d \rangle} \frac{E^3}{2\,\pi}.$$

Here $q(t)$ is a Fourier image of $q(\omega)$ in the time domain. In accordance with the Lyapunov theorem this proves stability of the stationary point of the Hamiltonian corresponding to its minimum. We would like to point out that this estimate does not prove stabilty of considered above DM soliton in strict mathematical sense, because one has to demonstrate that this mimimum is attained at analyzed solution. This requires more sophisticated mathematical treatment and will be published elsewhere. However, the boundedness of the Hamiltonian together with an observation that H is negative at the soliton solution is already a strong indication that DM soliton is stable in this region. We can easily get similar to [17, 18] an estimate on the maximal peak power of the pulse

$$\max(|q(t,z)|^2) \geq -\frac{4\,\pi\,H}{\epsilon\,E}, \quad \text{here} \quad H < 0. \tag{9}$$

Using arguments presented in [17] we conclude from Eq. (9) that any pulse with negative H cannot decay due to radiation of linear waves and will evolve to a state corresponding (in our case) to the minimum of the Hamiltonian. Note also that above result (boundedness of the Hamiltonian) is proved only for the region $\langle d \rangle > 0$. One can assume from this that though steady-state localized solutions of Eq. (6) do exist also in the region of the negative average dispersion, it is likely that their stability properties could be different from that in the region of $\langle d \rangle > 0$.

In conclusions, using path-averaged equation in the spectral domain we have studied analytically and numerically structure and dynamics of the chirped breathing soliton in the optical fiber lines. We have demonstrated that the Hamiltonian H is bounded from below and H is negative on the soliton solution. This proves stability of the stationary point of the Hamiltonian corresponding to this minimum.

We would like to thank E.Kuznetsov, I.Gabitov and N. Doran for valuable discussions. This research was supported in part by of RFBR grant # 96-02-19131-a).

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