

VERTEX OPERATORS FOR DEFORMED VIRASORO ALGEBRA

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Vertex operators for the deformed Virasoro algebra are defined, their bosonic representation is constructed and difference equation for the simplest vertex operators is described.

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1. *Introduction.* A feature characterising both classical and quantum integrable systems is the existence of the infinite dimensional Abelian symmetry. For two-dimensional lattice systems this symmetry appears as a family of commuting transfer matrices. From the standpoint of one-dimensional quantum chains this means that there are infinitely many mutually commuting integrals of motion.

There is a well known method for studying the eigenvalues of commuting transfer matrices. The Bethe Ansatz method makes it possible to obtain all eigenvalues and eigenvectors in a certain form, a using solution to a system of algebraic equations (the Bethe Ansatz equations). With this approach the main problem is to handle the Bethe Ansatz equations in the thermodynamic limit when the number of sites of the lattice tends to infinity.

Fortunately, in this limit solvable lattice models have another type of symmetry usually called dynamical one [1], [2]. For six-vertex model this symmetry is described by quantum affine group $U_q(\widehat{sl}_2)$ at the level one [1]. Similar picture appears when considering ABF model. In this case dynamical symmetry is described by Deformed Virasoro Algebra (DVA) [2].

Generators of the dynamical symmetry do not mutually commute; however they form the space of states of the model. The role of this symmetry in the lattice models is the same as that of the Virasoro algebra in Conformal Field Theory (CFT). Thus methods and results of CFT [3] may be used in attacking lattice models.

In CFT space of physical states is defined by irreducible representations of Virasoro algebra. Vertex operators intertwine this representations. Physically, operators correspond to primary fields of CFT. The overall space of states of the theory can be obtained by action of the vertex operators on the vacuum state. The bosonization is a powerful method of CFT [4, 5, 6]. It gives an explicit description of the space of states of the theory and enables one to calculate correlation functions of all physical fields of the theory.

In [7, 8] a bosonic representation for the DVA was described. A bosonic representation for the simplest vertex operators was given in [7]. Using this results, correlation functions of the ABF model were calculated in [2].

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In this study we give an abstract definition of vertex operators for the DVA and describe their bosonic representation. Our construction is in line with that of [6]. In conclusion we suggest an application of our construction to the fusion RSOS models [11] and discuss problems to be solved.

2. *Deformed Virasoro Algebra.* Let us recall some basic facts on the DVA [7, 8, 9]. The algebra depends on the two parameters $\xi > 0$ and $0 < x < 1$. The algebra is generated by the current $T(z) = \sum T_n z^{-n}$ with the following basic relations

$$f\left(\frac{w}{z}\right) T(z)T(w) - f\left(\frac{z}{w}\right) T(w)T(z) = \frac{(x^\xi - x^{-\xi})(x^{(\xi+1)} - x^{-(\xi+1)})}{x + x^{-1}} \left(\delta\left(\frac{w}{z}x^{-2}\right) - \delta\left(\frac{w}{z}x^2\right) \right), \quad (1)$$

where

$$f(z) = (1-z)^{-1} \frac{(zx^{2(\xi+1)}; x^4)_\infty (zx^{-2\xi}; x^4)_\infty}{(zx^{2(\xi+1)+2}; x^4)_\infty (zx^{-2\xi+2}; x^4)_\infty},$$

$$(z; q_1, \dots, q_p)_\infty = \prod_{n=0}^{\infty} (1 - zq_1^n) \dots (1 - zq_p^n)$$

and

$$\delta(z) = \sum_{m=-\infty}^{\infty} z^m$$

If we fix z and consider the limit as $x \rightarrow 1$, then

$$T(z) = 2 + \xi(\xi+1)(x - x^{-1})^2 \left(z^2 L(z) + \frac{1}{4\xi(\xi+1)} \right) + O((x - x^{-1})). \quad (2)$$

It can be verified that the defining relations (1) give Virasoro algebra commutators for the current $L(z)$ and that the corresponding central charge is

$$c = 1 - \frac{6}{\xi(\xi+1)} = 1 - 12\alpha_0^2, \quad (3)$$

where we introduce the notation similar to those used in CFT

$$\alpha_0 = \frac{1}{\sqrt{2\xi(\xi+1)}},$$

$$\alpha_+ = -\sqrt{\frac{2(\xi+1)}{\xi}},$$

$$\alpha_- = \sqrt{\frac{2\xi}{\xi+1}}. \quad (4)$$

The DVA may be bosonized in terms of deformed bosonic oscillators λ_m with the commutation relations

$$[\lambda_m, \lambda_n] = \frac{1}{m} \frac{(x^{\xi m} - x^{-\xi m})(x^{(\xi+1)m} - x^{-(\xi+1)m})}{x^m + x^{-m}} \delta_{n+m,0}. \quad (5)$$

We should also define "zero mode" operators P, Q which commute with λ_m and satisfy the relation:

$$[Q, P] = i. \quad (6)$$

The Heisenberg algebra (5), (6) is represented on the Fock space \mathcal{F}_p generated by the action of the creation operators λ_{-n} , $n > 0$, on the highest weight vector v_p

$$\lambda_n v_p = 0, \quad n \geq 0, \quad P v_p = p v_p. \quad (7)$$

Now we introduce the field

$$\Lambda(z) = x^{\sqrt{2\xi(\xi+1)}P} : \exp\left(-\sum_{m \neq 0} \lambda_m z^{-m}\right) :. \quad (8)$$

Then the current $T(z)$ may be written in the form [8], [10]

$$T(z) = x\Lambda(zx^{-1}) + x^{-1}\Lambda^{-1}(zx). \quad (9)$$

It can be easily verified that the field $T(z)$ satisfies (1).

In what follows of interest is a particular representations of DVA which correspond to minimal models of CFT [3]. This representations are parametrized by two positive coprime integer numbers p and p' ($p' > p$). In this parametrization we have

$$\xi = \frac{p}{p' - p}, \quad \alpha_+ = -\sqrt{2\frac{p'}{p}}, \quad \alpha_- = \sqrt{2\frac{p}{p'}}. \quad (10)$$

Let us consider the action of the DVA on the Fock spaces

$$\mathcal{F}_{l,k} = \mathcal{F}_{\alpha_l, k}, \quad (11)$$

where $\alpha_{l,k} = -\frac{1}{2}(\alpha_+ l + \alpha_- k)$, $k, l \in \mathbb{Z}$.

This action is highly reducible and some states should be excluded from the Fock module to obtain irreducible component. Explicitly, the procedure is as follows [2, 6]. Let us introduce "screening current"

$$I_+(z) = e^{i\alpha_+ Q} z^{\alpha_+ P + 1} : \exp \left\{ - \sum_{m \neq 0} \frac{x^m + x^{-m}}{x^{\xi m} - x^{-\xi m}} \lambda_m z^{-m} \right\} : \quad (12)$$

and the auxiliary function

$$F_+(z, P) = z^{-\alpha_+ P - \frac{\xi+1}{\xi}} x^{\xi \alpha_+ P(\alpha_+ P + 1) + \alpha_+ P} \frac{E(x^{-1-2\xi\alpha_+ P} z; x^{2\xi})}{E(x^{-1} z^{-1}; x^{2\xi})},$$

where

$$E(z; q) = (q; q)_\infty (z; q)_\infty (z^{-1} q; q)_\infty.$$

The "screening charge"

$$Q_+ = \oint dz I_+(z) F_+\left(\frac{z}{z_0}; P\right). \quad (13)$$

is of crucial importance in our analysis.

An important property of operator Q_+ is that it commutes with the DVA current $T(z)$ and does not depend on z_0 acting on the Fock space $\mathcal{F}_{l,k}$. We note that, in view of parametrization (10), $\alpha_{l,k} = \alpha_{l+p, k+p'}$; hence the Fock modules $\mathcal{F}_{l,k}$ and $\mathcal{F}_{l+p, k+p'}$ may be identified.

The screening charge (13) determines the following map:

$$Q_{2j} = Q_+^l : \mathcal{F}_{l-2jp, k} \rightarrow \mathcal{F}_{-l-2jp, k}, \quad (14)$$

$$Q_{2j+1} = Q_+^{p-l} : \mathcal{F}_{-l-2jp, k} \rightarrow \mathcal{F}_{l-2(j+1)p, k} \quad (15)$$

for $j \in \mathbb{Z}$, $1 \leq l \leq p-1$, $1 \leq k \leq p'-1$.

As a result, we can construct the infinite chain

$$\dots \xrightarrow{Q_{-2}} \mathcal{F}_{2p-l, k} \xrightarrow{Q_{-1}} \mathcal{F}_{l, k} \xrightarrow{Q_0} \mathcal{F}_{-l, k} \xrightarrow{Q_1} \dots$$

It was shown in [2] that this chain is the Felder resolution [6]. In other words

$$1. Q_j Q_{j-1} = 0$$

$$2. \text{Ker} Q_j / \text{Im} Q_{j-1} = \begin{cases} \mathcal{L}_{l, k}, & j = 0, \\ 0, & j \neq 0, \end{cases}$$

where $\mathcal{L}_{l, k}$ is an irreducible representation of the DVA.

3. *Vertex Operators for the DVA.* In the previous section we described the irreducible representations of the DVA. Now we construct vertex operators intertwining this representations. Our construction is parallel to that of [6]. At first we introduce second "screening charge"

$$Q_- = \oint dz I_-(z) F_-\left(\frac{z}{z_0}; P\right), \quad (16)$$

where

$$I_-(z) = e^{i\alpha-Q} z^{\alpha-P+1} : \exp \left\{ \sum_{m \neq 0} \frac{x^m + x^{-m}}{x^{(\xi+1)m} - x^{-(\xi+1)m}} \lambda_m z^{-m} \right\} : \quad (17)$$

and

$$F_-(z, P) = z^{-\alpha-P-\frac{\xi}{\xi+1}} x^{(\xi+1)\alpha-P(\alpha-P+1)-\alpha-P} \frac{E(x^{1-2(\xi+1)\alpha-P} z; x^{2\xi+2})}{E(xz^{-1}; x^{2\xi+2})}. \quad (18)$$

The operator Q_+ commutes with the DVA current $T(z)$ and the charge Q_- .

Then we introduce highest component of the vertex operator

$$V_{l,k} = e^{i\alpha_{l,k}Q} z^{\alpha_{l,k}P} : \exp \left\{ \sum_{m \neq 0} \frac{x^{\beta_{l,k}m} - x^{-\beta_{l,k}m}}{(x^{\xi m} - x^{-\xi m})(x^{(\xi+1)m} - x^{-(\xi+1)m})} \lambda_m z^{-m} \right\} :, \quad (19)$$

where $\beta_{l,k} = \sqrt{2\xi(\xi+1)}\alpha_{l,k} = (\xi+1)l - \xi k$, $1 \leq l \leq p-1$, $1 \leq k \leq pt-1$.

The other components of the vertex operator are given by the action of the screening charges on the highest component

$$\begin{aligned} V_{l,k}^{m,n}(z) &= \int V_{l,k}(z) \prod_{i=1}^m dw_i I_+(w_i) F_+\left(\frac{w_i}{z} x_{i-1,k}^\beta; P\right) \prod_{j=1}^n du_j I_-(u_j) F_-\left(\frac{u_j}{z} x^{-\beta_{l,k-1}}; P\right) \\ &= \int \prod_{i=1}^m dw_i F_+\left(\frac{z}{w_i} x_{i-1,k}^\beta; -P\right) I_+(w_i) \prod_{j=1}^n du_j F_-\left(\frac{z}{u_j} x^{-\beta_{l,k-1}}; -P\right) I_-(u_j) V_{l,k}(z), \end{aligned} \quad (20)$$

where $0 \leq m \leq l$, $0 \leq n \leq k$ and contours for w_i , u_j enclose the poles $w_i = zx^{-\beta_{l,k+1}+2\xi q}$, $u_j = zx^{\beta_{l+1,k}+(2\xi+2)q}$, ($q = 0, 1, 2, \dots$) respectively.

The vertex operator $V_{l,k}^{m,n}(z)$ maps Fock space $\mathcal{F}_{s,t}$ into $\mathcal{F}_{s+l-2m,t+k-2n}$.

Note that operators $V_{1,0}$ and $V_{0,1}$ coincide accurate to normalization factors with the vertex operators constructed in [2], [7]. In some sense these operators are fundamental, because any other vertex operator may be obtained as their product

$$V_{l,k}^{m,n}(z) =: \prod_{i=-l/2}^{l/2} V_{1,0}^{m_i,0} \left(x^{2i(\xi+1)} z \right) \prod_{j=-k/2}^{k/2} V_{0,1}^{0,n_j} \left(x^{2j\xi} z \right) : \quad (21)$$

here $m = \sum m_i$, $n = \sum n_j$.

To prove the intertwining property of the vertex operators we should test the following diagram

$$\begin{array}{ccc} \mathcal{F}_{s,t} & \xrightarrow{V_{l,k}^{m,n}(z)} & \mathcal{F}_{s+l-2m,t+k-2n} \\ \downarrow Q_+^t & & \downarrow Q_+^{t+k-2n} \\ \mathcal{F}_{s,-t} & \xrightarrow{V_{l,k}^{m,h-n}(z)} & \mathcal{F}_{s+l-2m,2n-t-k} \end{array}$$

for commutativity. This commutativity is a consequence of the following basic properties of the screening charges

$$\begin{aligned} Q_+^t V_{l,k}^{m,n}(z) &= (-1)^{t-l} V_{l,k}^{m,n+t}(z) \\ V_{l,k}^{m,n}(z) Q_+^t &= V_{l,k}^{m,n+t}(z). \end{aligned}$$

Thus operator (20) intertwines the irreducible representations of the DVA.

Using bosonic representation for $T(z)$ and $V_{l,k}^{m,n}(w)$ we may calculate commutation relations between them. We restrict our attention to the highest component $V_{l,k}(w)$ of vertex operator in view of the commutativity of $T(z)$ with the screening charges. Thus we obtain

$$\frac{w}{z} f_{l,k} \left(\frac{w}{z} \right) T(z) V_{l,k}(w) + f_{l,k} \left(\frac{z}{w} \right) V_{l,k}(w) T(z) =$$

$$\begin{aligned}
&= g_{l,k} \left\{ (wx^{-\xi-1})^{-\frac{\beta_{l-1,k}}{2\xi}} x \delta \left(\frac{w}{z} x^{\beta_{l,k}} \right) V_{l-1,k} (wx^{-\xi-1}) V_{1,0} (wx^{\beta_{l-1,k-2}}) + \right. \\
&+ (wx^{-\beta_{l-1,k-2}})^{-\frac{\beta_{l-1,k}}{2\xi}} x^{-1} \delta \left(\frac{w}{z} x^{-\beta_{l,k}} \right) V_{1,0} (wx^{-\beta_{l-1,k-2}}) V_{l-1,k} (wx^{\xi+1}) \left. \right\} = \\
&= g'_{l,k} \left\{ (wx^{\xi})^{\frac{\beta_{l,k-1}}{2\xi+2}} x \delta \left(\frac{w}{z} x^{\beta_{l,k}} \right) V_{l,k-1} (wx^{\xi}) V_{0,1} (wx^{-\beta_{l-2,k-1}}) + \right. \\
&+ (wx^{-\beta_{l-2,k-1}})^{\frac{\beta_{l,k-1}}{2\xi+2}} x^{-1} \delta \left(\frac{w}{z} x^{-\beta_{l,k}} \right) V_{0,1} (wx^{-\beta_{l-2,k-1}}) V_{l,k-1} (wx^{-\xi}) \left. \right\}, \quad (22)
\end{aligned}$$

where

$$\begin{aligned}
f_{l,k}(z) &= \frac{(zx^{\beta_{l,k}+2}; x^4)_{\infty} (zx^{-\beta_{l,k}+4}; x^4)_{\infty}}{(zx^{\beta_{l,k}}; x^4)_{\infty} (zx^{-\beta_{l,k}+2}; x^4)_{\infty}}, \\
g_{l,k} &= \frac{(x^{4\xi+4}; x^{2\xi}, x^4)_{\infty} (x^{2\beta_{l,k-1}}; x^{2\xi}, x^4)_{\infty}}{(x^{4\xi+2}; x^{2\xi}, x^4)_{\infty} (x^{2\beta_{l+1,k}}; x^{2\xi}, x^4)_{\infty}}
\end{aligned}$$

and

$$g'_{l,k} = \frac{(x^{-4\xi}; x^{-2(\xi+1)}, x^4)_{\infty} (x^{2\beta_{l-1,k}}; x^{-2(\xi+1)}, x^4)_{\infty}}{(x^{-4\xi-2}; x^{-2(\xi+1)}, x^4)_{\infty} (x^{2\beta_{l,k+1}}; x^{-2(\xi+1)}, x^4)_{\infty}}.$$

We may define a vertex operator as the intertwining operator for the representations of the DVA with defining relations (22).

It can be verified that, as $x \rightarrow 0$ our construction gives vertex operators for the Virasoro algebra. In particular, operator (20) gives the bosonic representation for the primary field with the conformal dimension

$$\Delta_{l,k} = \frac{((l-1)p - (k-1)p')^2 - (p-p')^2}{4pp'} \quad (23)$$

and equation (22) is rearranged to give

$$[L(z), V_{l,k}^{n,m}(w)] = z^{-1} \delta \left(\frac{w}{z} \right) \frac{\partial}{\partial w} V_{l,k}^{n,m}(w) + z^{-2} \Delta_{l,k} \delta \left(\frac{w}{z} \right) V_{l,k}^{n,m}(w). \quad (24)$$

which coincides with defining relations for a primary field in CFT.

4. *Conclusion.* In the previous sections we defined vertex operators for the DVA and constructed their bosonic representations. Now let us review some of our further results and problems.

1. In CFT Fock module $\mathcal{F}_{1,0}$ contains a cosingular vector at the second level. This results in the differential equation of the second order for the vertex operator $V_{1,0}(z)$

$$\partial^2 V_{1,0}(w) = \frac{3}{4\Delta_{1,0} + 2} \oint_w \frac{L(z) V_{1,0}(w)}{z-w} dz. \quad (25)$$

Using this equation and the Ward identities we can derive differential equations for correlation functions in CFT.

A similar situation emerges in the DVA. In this case the vertex operator $V_{1,0}(z)$ satisfies difference equation of the second order

$$\begin{aligned}
&\mathcal{D}_{x^{2\xi}}^2 V_{1,0}(w) = \\
&= \frac{1}{(x^{\xi} - x^{-\xi})^2} \left(\oint_w z^{-3} f_{1,0} \left(\frac{w}{z} \right) T(z) V_{1,0}(w) dz - (x^{\xi} - x^{-\xi}) w^{-2} V_{1,0}(w) \right). \quad (26)
\end{aligned}$$

Up to now, we do not know the Ward identities for the DVA, and it is unclear how to obtain a difference equation for correlation functions of the DVA.

2. In [2] the vertex operator $V_{0,1}(z)$ was interpreted as a semi-infinite transfer matrix for the ABF model (in this case we should put $p' = p + 1$). Using the

coner transfer matrix method and bosonic representation for the vertex operator correlation, functions of the ABF model in the regime III were calculated. From representation (21) and exchange relations for operators $V_{0,1}(z)$ [2] it follows that operators $V_{0,n}(z)$ give semi-infinite transfer matrices for the fusion RSOS models described in [11]. The bosonic representation (20) makes it possible to calculate correlation functions of the fusion RSOS models following the lines of [2]. We will concentrate on this problem in a futher publications.

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