FULLY QUANTUM TREATMENT OF THE LANDAU–POMERANCHIK–MIGDAL EFFECT IN QED AND QCD

B.G.Zakharov

L.D.Landau Institute for Theoretical Physics RAS

117334 Moscow, Russian Federation

For the first time a rigorous quantum treatment of the Landau–Pomeranchuk–Migdal effect in QED and QCD is given. The rate of photon (gluon) radiation by an electron (quark) in medium is expressed through the Green's function of a two-dimensional Schrödinger equation with an imaginary potential. In QED this potential is proportional to the dipole cross section for scattering of e^+e^- pair off an atom, in QCD it is proportional to the cross section of interaction with color centre of the color singlet quark-antiquark-gluon system.

PACS: 13.60.Le, 12.38.Qk, 25.20.Lj

The effect of multiple scattering on bremsstrahlung (the Landau-Pomeranchuk-Migdal (LPM) effect [1, 2]) in QED and QCD has recently attracted much attention [3, 4]. However, a rigorous treatment of the LPM effect for an arbitrary energy of the photon (gluon) is as yet lacking. In the present paper, based on the technique of Ref. [5], we develop a quantum theory of the LPM effect for the whole range of photon (gluon) energies. As usual, we treat the medium as a system of uncorrelated static scatterers (atoms).

In order to set the background we start with the LPM effect in QED. In vacuum the evolution of the wave function without radiative corrections of a relativistic electron with longitudinal momentum p_z in the variable $\tau = (t+z)/2$ at $(t-z) =$ const is described by the Hamiltonian $H = (p^2 + m_e^2)/2\mu_e$ [6], where p is a transverse momentum, m_e is the electron mass, and $\mu_e = p_z$. Consequently, the electron wave functions at planes $z = z_1$ and $z = z_2$ are related by

$$\psi(\rho_2, z_2) = \int d\rho_1 K_e(\rho_2, z_2|\rho_1, z_1)\psi(\rho_1, z_1),\tag{1}$$

where $\rho_{1,2}$ are the transverse coordinates, and K_e is the Green's function of the two-dimensional Schrödinger equation for a particle with mass μ_e, times the phase factor $\exp[-im_e^2(z_2 - z_1)/2\mu_e]$. Eq. (1) holds for each helicity state. At high energy spin effects in interaction of an electron with an atom vanish, and equation analogous to (1) holds for propagation of an electron through medium as well. The corresponding propagator reads

$$K_e(\rho_2, z_2|\rho_1, z_1) = \int \mathcal{D}\rho \exp\left\{i \int dz \left[\frac{\mu_e \dot{\rho}^2}{2} + e\, U(\rho, z)\right] - \frac{im_e^2(z_2 - z_1)}{2\mu_e}\right\},\tag{2}$$

where $\dot{\rho} = d\rho/dz$, $U(\rho, z)$ is the potential of the medium.

The interaction of an electron with the photon field generates the radiative correction, δK_e, to the propagator (2). To leading order in $\alpha = 1/137$, δK_e is generated by sequential transitions $e \to e'\gamma \to e$. The probability of passage of

an electron through the target without radiation of the photon can be written in terms of K_e and δK_e as

$$P_e = 1 + 2\text{Re} \int d\rho_1 d\rho'_1 d\rho_2 \left[\langle \delta K_e(\rho_2, z_2|\rho_1, z_1) K_e^*(\rho_2, z_2|\rho'_1, z_1) \rangle - (\text{vac}) \right] , \qquad (3)$$

where $\langle ... \rangle$ means averaging over the states of the target, and (vac) denotes the vacuum $\delta K_e K_e^*$ term. The subtraction of the vacuum term takes into account the renormalization of the electron wave function. Eq. (3) is written for the target with the density independent of ρ, and the points z_1 and z_2 are assumed to be at large distances before and after the target, respectively. The initial electron flux is normalized to unity. Evaluation of P_e allows to determine the probability of radiation of the photon, P_γ, through the unitarity relation $P_e + P_\gamma = 1$.

The contribution of transitions $e \to e'\gamma \to e$ to δK_e can be written as

$$\delta K_e(\rho_2, z_2|\rho_1, z_1) = - \int_0^1 dx \int_{z_1}^{z_2} d\xi_1 \int_{\xi_1}^{z_2} d\xi_2 \int d\tau_1 d\tau_2 g(\xi_1, \xi_2, x) \times$$

$$\times K_e(\rho_2, z_2|\tau_2, \xi_2) K_{e'}(\tau_2, \xi_2|\tau_1, \xi_1) K_\gamma(\tau_2, z'_2|\tau_1, \xi_1) K_e(\tau_1, \xi_1|\rho_1, z_1) . \qquad (4)$$

Here the indices e' and γ label the electron and photon propagators for the intermediate $e'\gamma$ state. The transverse masses which enter $K_{e'}$ and K_γ are given by $\mu_{e'} = (1-x)\mu_e$ and $\mu_\gamma = x\mu_e$, where x is the light-cone Sudakov variable of the photon. The vertex function $g(\xi_1, \xi_2, x)$ equals

$$g(\xi_1, \xi_2, x) = \Lambda_{nf}(x)[\mathbf{v}_\gamma(\xi_2) - \mathbf{v}_{e'}(\xi_2)] \cdot [\mathbf{v}_\gamma(\xi_1) - \mathbf{v}_{e'}(\xi_1)] + \Lambda_{sf}(x) , \qquad (5)$$

where $\Lambda_{nf}(x) = \alpha[4 - 4x + 2x^2]/4x$, $\Lambda_{sf}(x) = \alpha m_e^2 x/2\mu_{e'}^2$, $\mathbf{v}_\gamma$ and $\mathbf{v}_{e'}$ are the transverse velocity operators, which act on the corresponding propagators in Eq. (4). The two terms in (5) correspond to the $e \to e'\gamma$ transitions conserving (nf) and changing (sf) the electron helicity.

Making use of Eqs. (2)–(4) we can write the rate of the bremsstrahlung in the following differential form (we suppress the vacuum term, which will be recovered in the final formula (15))

$$\frac{dP_\gamma}{dx} = 2\text{Re} \int_{z_1}^{z_2} d\xi_1 \int_{\xi_1}^{z_2} d\xi_2 \int d\rho_1 d\rho'_1 d\tau_1 d\tau'_1 d\tau_2 d\tau'_2 d\rho_2 g(\xi_1, \xi_2, x) \times$$

$$\times S(\rho_2, \rho'_2, z_2|\tau_2, \tau'_2, \xi_2) M(\tau_2, \tau'_2, \xi_2|\tau_1, \tau'_1, \xi_1) S(\tau_1, \tau'_1, \xi_1|\rho_1, \rho'_1, z_1) , \qquad (6)$$

where S and M are defined as

$$S(\rho_2, \rho'_2, \xi_2|\rho_1, \rho'_1, \xi_1) = \int \mathcal{D}\rho_e \mathcal{D}\rho'_e \exp \left[\frac{i\mu_e}{2} \int d\xi (\dot{\rho}_e^2 - \dot{\rho}_e'^2) \right] \Phi(\{\rho_e\}, \{\rho'_e\}) , \qquad (7)$$

$$M(\rho_2, \rho'_2, \xi_2|\rho_1, \rho'_1, \xi_1) = \int \mathcal{D}\rho_e \mathcal{D}\rho_{e'} \mathcal{D}\rho_\gamma \exp \left\{ \frac{i}{2} \int d\xi (\mu_{e'}\dot{\rho}_e^2 + \mu_\gamma \dot{\rho}_\gamma^2 - \mu_e \dot{\rho}_e'^2) - \right.$$

$$\left. - \frac{i(\xi_2 - \xi_1)}{l_f} \right\} \Phi(\{\rho_{e'}\}, \{\rho_e\}) , \qquad (8)$$

$$\Phi(\{\rho_i\}, \{\rho_j\}) = \left\langle \exp\left\{ ie \int d\xi [U(\rho_i(\xi), \xi) - U(\rho_j(\xi), \xi)] \right\} \right\rangle . \tag{9}$$

Here $l_f = 2\mu_e(1-x)/m_e^2 x$ is usually called the photon formation length. The boundary conditions for trajectories in Eq. (7) are $\rho_e(\xi_{1,2}) = \rho_{1,2}$, $\rho'_e(\xi_{1,2}) = \rho'_{1,2}$, and in Eq. (8) $\rho_{e',\gamma}(\xi_{1,2}) = \rho_{1,2}$, $\rho_e(\xi_{1,2}) = \rho'_{1,2}$. Averaging over positions of atoms in Eq. (9) yields [5]

$$\Phi(\{\rho_i\}, \{\rho_j\}) = \exp\left[-\frac{1}{2} \int d\xi n(\xi)\sigma(|\rho_i - \rho_j|) \right] , \tag{10}$$

here $n(\xi)$ is the target density, and $\sigma(\rho)$ is the dipole cross section for scattering of the e^+e^- pair of transverse size ρ on a free atom. For the atomic potential $\phi(r) = 4\pi(Z\alpha/r)\exp(-a/r)$ $(a = 1.4Z^{-1/3}r_B)$ $\sigma(\rho)$ is given by

$$\sigma(\rho) = 8\pi(Z\alpha a)^2 \left[1 - \frac{\rho}{a} K_1\left(\frac{\rho}{a}\right) \right] . \tag{11}$$

For $\rho \ll a$, which will be important in the considered problem, $\sigma(\rho) \simeq C(\rho)\rho^2$, where

$$C(\rho) = 4\pi(Z\alpha)^2 \left[\log\left(\frac{2a}{\rho}\right) + \frac{(1 - 2\gamma)}{2} \right] , \qquad \gamma = 0.577 . \tag{12}$$

For nuclei of finite radius R_A, Eq. (12) holds for $\rho \gtrsim R_A$, and $C(\rho \lesssim R_A) = C(R_A)$.
Analytical evaluation of the path integral (7) yields [5]

$$S(\rho_2, \rho'_2, \xi_2|\rho_1, \rho'_1, \xi_1) = \left(\frac{\mu_e}{2\pi\Delta\xi}\right)^2 \exp\left\{ \frac{i\mu_e}{2\Delta\xi}[(\rho_1 - \rho_2)^2 - (\rho'_1 - \rho'_2)^2] - \right.$$
$$\left. - \frac{1}{2} \int d\xi n(\xi)\sigma(|\tau_s(\xi)|) \right\} , \tag{13}$$

$$\tau_s(\xi) = (\rho_1 - \rho'_1)\frac{(\xi_2 - \xi)}{\Delta\xi} + (\rho_2 - \rho'_2)\frac{(\xi - \xi_1)}{\Delta\xi} , \qquad \Delta\xi = \xi_2 - \xi_1 .$$

Introducing in (8) the Jacobi variables $\alpha = (\mu_{e'}\rho_{e'} + \mu_\gamma\rho_\gamma)/(\mu_{e'} + \mu_\gamma)$ and $\beta = \rho_{e'} - \rho_\gamma$, and integrating over the trajectories $\alpha(\xi)$ and $\rho_e(\xi)$ one can obtain

$$M(\rho_2, \rho'_2, \xi_2|\rho_1, \rho'_1, \xi_1) = \left(\frac{\mu_e}{2\pi\Delta\xi}\right)^2 \exp\left\{ \frac{i\mu_e}{2\Delta\xi}[(\rho_1 - \rho_2)^2 - (\rho'_1 - \rho'_2)^2] - \frac{i\Delta\xi}{l_f} \right\} \times$$
$$\times \int \mathcal{D}\beta \exp\left\{ i \int d\xi \left[\frac{\mu_{e'\gamma}\dot{\beta}^2}{2} + i\frac{n(\xi)\sigma(|\tau_m(\xi)|)}{2} \right] \right\} , \tag{14}$$

$$\tau_m(\xi) = (\rho_1 - \rho'_1)\frac{(\xi_2 - \xi)}{\Delta\xi} + (\rho_2 - \rho'_2)\frac{(\xi - \xi_1)}{\Delta\xi} + \frac{\beta(\xi)\mu_\gamma}{(\mu_{e'} + \mu_\gamma)} ,$$

where $\mu_{e'\gamma} = \mu_{e'}\mu_\gamma/(\mu_{e'} + \mu_\gamma) = E_e x(1 - x)$ (E_e is the electron energy).
Substituting (13), (14) into (6), and integrating in (6) over the transverse variables we finally obtain (we set $-z_1 = z_2 = \infty$, and recover the vacuum term)

$$\frac{dP_\gamma}{dx} = 2\text{Re} \int_{-\infty}^{\infty} d\xi_1 \int_{\xi_1}^{\infty} d\xi_2 \exp\left(-\frac{i\Delta\xi}{l_f}\right) g(\xi_1, \xi_2, x) \left[K_{e'\gamma}(0, \xi_2|0, \xi_1) - K_{e'\gamma}(0, \xi_2|0, \xi_1)|_{n=0} \right] ,$$
$$\tag{15}$$

$$K_{e'\gamma}(\beta_2, \xi_2 | \beta_1, \xi_1) = \int \mathcal{D}\beta \exp\left\{ i \int d\xi \left[\frac{\mu_{e'\gamma}\dot{\beta}^2}{2} - v(\beta, \xi) \right] \right\}, \qquad (16)$$

$$v(\beta, \xi) = -i \frac{n(\xi)\sigma(|\beta|x)}{2}. \qquad (17)$$

Thus, we expressed the intensity of the photon radiation through the Green's function of the Schrödinger equation with the imaginary potential (17). Equation (15) is the main result of the present paper.

To proceed with analytical evaluation of the radiation density we take advantage of the slow β-dependence of $C(|\beta|x)$ at $|\beta|x \lesssim 1/m_e$ which as we will show below are important in Eq. (15). Evidently, to logarithmic accuracy we can replace (17) by the harmonic oscillator potential with the frequency

$$\Omega = \frac{(1-i)}{\sqrt{2}} \left(\frac{nC(\beta_{eff}x)x^2}{\mu_{e'\gamma}} \right)^{1/2} = \frac{(1-i)}{\sqrt{2}} \left(\frac{nC(\beta_{eff}x)x}{E_e(1-x)} \right)^{1/2}. \qquad (18)$$

Here β_{eff} is the typical value of $|\beta|$ for trajectories contributing to the radiation density. Making use of the oscillator Green's function, after some algebra one can obtain from Eq. (15) the intensity of bremsstrahlung per unit length in the infinite medium

$$\frac{dP_\gamma}{dxdL} = n \left(\frac{C(\beta_{eff}x)}{C(1/m_e)} \right) \left[\left(\frac{d\sigma}{dx} \right)^{BH}_{nf} S_{nf}(\eta) + \left(\frac{d\sigma}{dx} \right)^{BH}_{sf} S_{sf}(\eta) \right]. \qquad (19)$$

Here $\eta = l_f |\Omega|$, and $(d\sigma/dx)^{BH}_{nf,sf}$ are the Bethe–Heitler cross sections conserving and changing the electron helicity. The factors S_{nf}, S_{sf} are given by

$$S_{nf}(\eta) = \frac{3}{\eta\sqrt{2}} \int_0^\infty dy \left(\frac{1}{y^2} - \frac{1}{\text{sh}^2 y} \right) \exp\left(-\frac{y}{\eta\sqrt{2}} \right) \left[\cos\left(\frac{y}{\eta\sqrt{2}} \right) + \sin\left(\frac{y}{\eta\sqrt{2}} \right) \right], \quad (20)$$

$$S_{sf}(\eta) = \frac{6}{\eta^2} \int_0^\infty dy \left(\frac{1}{y} - \frac{1}{\text{sh}y} \right) \exp\left(-\frac{y}{\eta\sqrt{2}} \right) \sin\left(\frac{y}{\eta\sqrt{2}} \right). \qquad (21)$$

At small η $S_{nf} \simeq 1 - 16\eta^4/21$, and $S_{sf} \simeq 1 - 31\eta^4/21$. Up to the slowly dependent on η factor $C(\beta_{eff}x)/C(1/m_e)$, the suppression of bremsstrahlung at $\eta \gg 1$ is controlled by the asymptotic behavior of the factors (20), (21): $S_{nf} \approx 3/\eta\sqrt{2}$, $S_{sf} \approx 3\pi/2\eta^2$. Eqs. (20), (21) allow to estimate β_{eff}. The variable of integration in (20), (21) in terms of $\Delta\xi$ in Eq. (15) equals $|\Delta\xi\Omega|$. Therefore, for typical values of $\Delta\xi$ contributing to the integral (15), $\Delta\xi_{eff}$, we have $\Delta\xi_{eff} \sim \min(l_f, 1/|\Omega|)$. Then, β_{eff} can be estimated from the obvious Schrödinger diffusion relation $\beta_{eff} \sim (2\Delta\xi_{eff}/\mu_{e'\gamma})^{1/2}$. In the limit of low density, when $\eta \to 0$, we get from this relation $\beta_{eff} \sim 1/m_e x$, and the right-hand side of (19) reduces to the Bethe–Heitler cross section times the target density. In the soft photon limit ($x \to 0$) at n fixed, Eqs. (19)-(21) yield

$$\frac{dP_\gamma}{dxdL} = 2\alpha^2 Z \sqrt{\frac{2n \log(2a/\beta_{eff}x)}{\pi E_e x}}, \qquad (22)$$

where $\beta_{eff} \sim \left[\pi(Z\alpha)^2 n E_e x^3 \log(2/\alpha Z^{1/3})\right]^{-1/4}$. Notice, that (22) differs from the prediction of Ref. [2] which corresponds to a replacement of $\beta_{eff}x$ by R_A in (22).

Let us now briefly consider a gluon bremsstrahlung by a quark interacting with the color screened Coulomb centres. Following Refs. [3, 4] we introduce a gluon mass $m_g = 1/a_{QCD}$, and treat the interaction of a quark with each centre in the Born approximation. The derivation of the gluon radiation rate follows closely the QED case. Making use of the fact that $-T_q^* = T_{\bar{q}}$ (here $T_{q,\bar{q}}$ are the color generators for quark and antiquark) and of the closure over final states of color centres, one can easily show that in QCD one should only to replace in (17) the QED dipole cross section $\sigma(|\beta|x)$ by the QCD three-body cross section of scattering of the color singlet $q\bar{q}g$ system on the color centre, σ_3 [7]. The latter in terms of the color dipole cross section for scattering of a color singlet $q\bar{q}$ pair, σ_2, is given by

$$\sigma_3(|\beta|) = \frac{9}{8}\left[\sigma_2(|\beta|) + \sigma_2(|\beta|(1-x))\right] - \frac{1}{8}\sigma_2(|\beta|x). \qquad (23)$$

For the formation length in QCD we have $l_f = 2E_q x(1-x)/[m_q^2 x^2 + m_g^2(1-x)]$. The oscillator approximation is applicable for evaluation of the radiation density at $\eta_{QCD} \gg 1$. In this regime $\beta_{eff} \ll a_{QCD}$, and to logarithmic accuracy the intensity of gluon bremsstrahlung can be evaluated making use of the oscillator Green's function with the oscillator frequency

$$\Omega_{QCD} = \frac{(1-i)}{\sqrt{2}}\left(\frac{nC_3(\beta_{eff},x)}{\mu_{q'g}}\right)^{1/2} = \frac{(1-i)}{\sqrt{2}}\left(\frac{nC_3(\beta_{eff},x)}{E_q x(1-x)}\right)^{1/2}. \qquad (24)$$

Here

$$C_3(\beta_{eff},x) = \frac{9}{8}\left[C_2(\beta_{eff}) + C_2(\beta_{eff}(1-x))\right] - \frac{1}{8}C_2(\beta_{eff}x),$$

and $C_2(\rho) \simeq (\alpha_S^2 \pi C_T/6) \log(2a_{QCD}/\rho)$ (C_T is the second order Casimir for the color centre).

In QCD the limit $x \to 0$ at the same time corresponds to $\eta \to 0$. Thus at sufficiently small x the Bethe–Heitler regime takes place. On the other hand, at $\eta_{QCD} \gg 1$ and $x \ll 1$ the oscillator approximation yields

$$\frac{dP_g}{dx dL} = 2\alpha_S^2 \sqrt{\frac{nC_T \log(2a_{QCD}/\beta_{eff})}{3\pi E_q x^3}}, \qquad (25)$$

with $\beta_{eff} \sim [x\alpha_S^2 C_T n E_q]^{-1/4}$. Eq. (25) disagrees with $dP_g/dx dL \propto x^{-3}$ predicted in Ref. [4].

1. L.D.Landau and I.Ya.Pomeranchuk, Dokl. Akad. Nauk SSSR **92**, 535, 735 (1953).
2. A.B.Migdal, Phys. Rev. **103**, 1811 (1956).
3. M.Gyulassy and X.-N. Wong, Nucl. Phys. **B420**, 583 (1994); X.-N.Wong, M.Gyulassy, and M.Plümer, Phys. Rev. **D51**, 3436 (1995).
4. R.Baier, Yu.L.Dokshitzer, S.Peigne, and D.Schiff, Phys. Lett. **B345**, 277 (1995).
5. B.G.Zakharov, Sov. J. Nucl. Phys. **46**, 92 (1987).
6. J.M.Bjorken, J.B.Kogut, and D.E.Soper, Phys. Rev. **D3**, 1382 (1971).
7. N.N.Nikolaev, G.Piller, and B.G. Zakharov, JETP **81**, 851 (1995).