Pis'ma v ZhETF, vol.64, iss.5, pp.355 - 360 © 1996 September 10

SPECTRAL RIGIDITY AND EIGENFUNCTION CKRREKATIONS AT THE ANDERSON TRANSITION

J.T.Chalker, V.E.Kravtsov[+], *I.V.Lerner*[*]

Theoretical Physics, University of Oxford
Oxford OX1 3NP, United Kingdom

[+] *International Center for Theoretical Physics*
34100 Trieste, Italy

Landau Institute for Theoretical Physics
117940 Moscow, Russia

[*] *School of Physics and Space Research*
University of Birmingham, Edgbaston, United Kingdom

Submitted 25 July 1996
Resubmitted 12 August 1996

The statistics of energy levels for a disordered conductor are considered in the critical energy window near the mobility edge. It is shown that, if critical wave functions are multifractal, the one-dimensional gas of levels on the energy axis is *compressible*, in the sense that the variance of the level number in an interval is $\langle(\delta N)^2\rangle \sim \chi\langle N\rangle$ for $\langle N\rangle \gg 1$. The compressibility, $\chi = \eta/2d$, is given *exactly* in terms of the multifractal exponent $\eta = d - D_2$ at the mobility edge in a d-dimensional system.

PACS: 05.45+b, 71.25.-s, 72.15.Rn.

An exact determination of energy levels in complex quantum systems is neither possible, nor desirable. Research in this area has focused instead on a statistical description of the spectra of such systems, which include: compound nuclei, quantum systems whose classical behaviour is chaotic, and mesoscopic disordered conductors. In many cases, statistical characteristics turn out to be universal, i.e. independent of most microscopic details of the system and thus common to different systems which belong to the same universality class. In the following we shall be concerned with the specific example of a disordered conductor at an Anderson transition, for which universal spectral properties are an aspect of critical behaviour at the transition. For weakly disordered conductors, where wavefunctions corresponding to different levels have spatial overlap, there exists repulsion between the levels close in energy. Typically such repelling levels are governed [1–3] by Wigner–Dyson statistics (WDS) [4] which describe correlations between the eigenvalues in random-matrix ensembles of certain symmetries [5]. At the opposite extreme of strong disorder, spatial overlap between wavefunctions corresponding to different levels is absent, the levels are uncorrelated and hence governed by Poisson statistics (PS).

Repulsion between neighbouring levels is only one of the statistical characteristics which are totally different in WDS and PS. A second is spectral rigidity, which characterises fluctuations in the level density on scales large compared to the mean spacing. Specifically, consider the variance, $\Sigma_2(\langle N\rangle) \equiv \langle N^2\rangle - \langle N\rangle^2$, for level number fluctuations in an energy interval which on average contains a large number, $\langle N\rangle$, of levels. Spectral rigidity is absent for PS: $\Sigma_2(\langle N\rangle) = \langle N\rangle$, the maximal possible value. On the other hand, level fluctuations are strongly suppressed for WDS:

$\Sigma_2(\langle N\rangle) \sim \ln\langle N\rangle$ [5]; the levels are almost rigid. It is useful to think of the energy levels as the coordinates of a one-dimensional gas on the energy axis. If this gas has compressibility χ, one expects $\Sigma_2(\langle N\rangle) \sim \chi\langle N\rangle$ for $\langle N\rangle \gg 1$. More precisely, we define (for system size L)

$$\chi = \lim_{\langle N\rangle\to\infty} \lim_{L\to\infty} \frac{d\Sigma_2(\langle N\rangle)}{d\langle N\rangle}. \tag{1}$$

Clearly, $\chi = 1$ for a system of uncorrelated compressible levels (PS), and $\chi = 0$ for a system of rigid, incompressible levels (WDS).

The subject we consider here is the existence of an intermediate *universal* limit for the compressibility (1). Spectral statistics in many systems experience a crossover from WDS to PS as some parameter of the system is changed. A smooth crossover of this type takes place for spectral statistics of electrons in $2d$ or quasi $1d$ disordered conductors with increasing disorder. A new statistical regime [6,7] arises for $d > 2$ at the critical value of disorder that corresponds to the Anderson metal-insulator transition. The value of the spectral compressibility at the transition was first considered in Ref. [8] and has been a subject of controversy [7–10]. Note, however, that all the previous discussion has been limited to the framework of one-parameter scaling.

We will show here that the one-parameter scaling is insufficient for the description of level statistics in the vicinity of the Anderson transition. The main result of the present work is an exact expression for the spectral compressibility at a mobility edge:

$$\chi = \frac{\eta}{2d}. \tag{2}$$

Here $\eta \equiv d - D_2$ is one of the exponents characterising properties of the critical eigenstates: the set of multifractal dimensions, D_p, govern the scaling behaviour of the moments of inverse participation ratio,

$$\left\langle \int d^d r |\psi_n(r)|^{2p} \right\rangle \propto L^{-D_p(p-1)}, \tag{3}$$

for integer p. It is only the fractal dimensionality D_2 that enters Eq. (2), because the compressibility depends only on the two-level correlation function (TLCF), which (as we show below) can be expressed solely via second-order wavefunction correlations.

Now we turn to the derivation of the result, Eq. (2). First, we express χ via the form factor of the TLCF. Then we outline a non-perturbative approach to calculating the form factor, developed previously in Refs. [11,12], which relates the spectral and wavefunction correlations. Finally, in the critical regime near the Anderson transition we relate the form factor to multifractal properties of the wavefunctions.

We define the TLCF as

$$R(s) = \langle\rho\rangle^{-2}\Big\{\big\langle\rho(\varepsilon + s\Delta)\,\rho(\varepsilon)\big\rangle\Big\} - 1, \tag{4}$$

where $\rho(\varepsilon) = L^{-d}\sum_n \delta(\varepsilon - E_n)$ is the density of states per unit volume, L is the sample size, and E_n are eigenvalues of the Hamiltonian for a system of free electrons in a random potential. The mean density of states, $\langle\rho\rangle$, varies only on the scale of the Fermi energy, ε_F. Thus the mean level spacing $\Delta = 1/(\langle\rho\rangle L^d)$ is considered constant.

The number variance in an energy window of width $E \equiv \langle N\rangle\Delta \ll \varepsilon_F$ is given in terms of the TLCF by $\Sigma_2(\langle N\rangle) = \int_{-\langle N\rangle}^{\langle N\rangle} (\langle N\rangle - |s|)\,R(s)\,ds$. Differentiating with

respect to $\langle N \rangle$, and taking the limit $\langle N \rangle \to \infty$ *after* the thermodynamic limit, $L \to \infty$, one obtains [7]

$$\chi = \int_{-\infty}^{+\infty} R(s)\, ds \equiv \lim_{t \to 0} K(t)\,, \tag{5}$$

where we have introduced the spectral form factor of the TLCF:

$$K(t) = \int_{-\infty}^{\infty} e^{-is t/t_H} R(s)\, ds\,, \tag{6}$$

and $t_H \equiv \hbar/\Delta$ is the Heisenberg time. It is, of course, only in the thermodynamic limit that one can expect universal behaviour. Taking this limit at fixed $\langle N \rangle$ leaves Δ and t_H, respectively, as the sole energy and time scales. Note that, for a system at a mobility edge, the number of critical levels (those for which the localisation length is bigger than the system size) diverges in the thermodynamic limit, as is required if there is to be a possibility that χ takes a non-trivial critical value. We comment in passing that a well-know sum rule, $\int_{-\infty}^{+\infty} R(s)\, ds = 0$, can be derived if one takes limits in the opposite order: $\langle N \rangle \to \infty$ *before* the thermodynamic limit, $L \to \infty$. The fact that the order of limits does not necessarily commute has been discussed extensively elsewhere [9,10,13].

To obtain χ, we use a recently developed [11,12] non-perturbative approach to calculating $K(t)$ for times $t \ll t_H$. In this approach, $K(t)$ is given by

$$K(t) = \frac{1}{2}\, \frac{|t|\, p(t)}{\pi\hbar\rho + \int_{0+}^{|t|} p(t')dt'}\,. \tag{7}$$

Here $p(t) = \langle |\Psi(0,t)|^2 \rangle$ is the ensemble-averaged quantum return probability for a wavepacket,

$$\Psi(r,t) = \sqrt{V_0} \sum_n \psi_n^*(0)\psi_n(r) e^{-iE_n t/\hbar},$$

that initially occupies a small volume V_0 near the origin[1)] The initial condition implies that the summation here is limited to the number of levels $\mathcal{N}_0 \sim L^d/V_0$ with energies lying within the energy band of width $E_0 \sim 1/\rho V_0$. Using spatial and spectral homogeneity, one can show that

$$p(t) = \left\langle \sum_{l<\mathcal{N}_0} c_{n,n+l} e^{-i(E_n - E_{n+l})t/\hbar} \right\rangle, \quad c_{n,m} \equiv \int d^d r\, |\psi_n(r)|^2 |\psi_m(r)|^2\,. \tag{8}$$

Thus $p(t)$ and ultimately $K(t)$, Eq. (7), are related to the local wavefunction correlations.

Before applying Eq. (7) to find χ, let us outline its derivation. A central idea is to consider the dependence of energy levels on some external parameter, λ. Then, Eq. (7) can be obtained in two ways [11,12]. The first is phenomenological: we consider [11] a random walk in the ensemble of impurity configurations parametrised by λ. Such a Brownian-motion level dynamics in the fictitious time $\propto \lambda^2$ is similar to that originally introduced by Dyson [14] in the context of random matrix theory. These dynamics generate the level correlations given by Eq. (7). Alternatively [12], using the homogeneity of level correlations at different points of the parametric space, one can prove for $t \neq 0$ the identity

$$tp(t) = \int_0^t dt' \left\langle \sum_{lm} c_{n,n+l}\, e^{i(E_n - E_{n+l})t'/\hbar}\, e^{-i(E_n - E_{n+m})t/\hbar} \right\rangle, \tag{9}$$

[1)] It is convenient to choose $V_0 \sim \ell^d$, where ℓ is the elastic mean free path.

which relates $p(t)$ to higher-order correlations between energy levels and wavefunctions (we stress that in constrast to Eq. (8), the summation here is extended over *all* eigenvalues of the Hamiltonian).

In the limit $t \ll t_H$ it is possible to factorise the average on the r.h.s. of Eq. (9). Consider first the sum over l. In both the metallic and the critical regimes, wavefunctions corresponding to different energies have spatial overlap, and so $c_{n,n+l}$ is a slowly decreasing function of $|l|$. Thus the oscillating factor, $e^{-i(E_n - E_{n+l})t'/\hbar}$, acts as a cut-off for the sum over l: the number of terms contributing is of order $t_H/t' > t_H/t \gg 1$. As a result, in the limit $t \to 0$, the entire contribution comes from terms for which $|l| \gg 1$. Now suppose that for one of these terms (i.e. fixed l, with $|l| \gg 1$) we carry out the sum on m. The result will depend on the conditional distribution of E_{n+m}, given that there are levels at the energies E_n and E_{n+l}. Moreover, for $t \neq 0$, non-zero contributions arise only from energy regions in which E_{n+m} has a non-uniform distribution due to correlations with E_n and E_{n+l}. The energy scale for such correlations is set only by Δ, and so the contribution to Eq.(10) comes from the two regions: $|E_{n+m} - E_n| \leq \Delta$ and $|E_{n+m} - E_{n+l}| \leq \Delta$. Both sets of correlations make equal contributions to the r.h.s. of Eq. (9), as can be seen by changing variable in the integral from t' to $t''=t-t'$. Taking this into account, we factorise the average in Eq. (9) as

$$tp(t) = 2 \left\langle \sum_m e^{-i(E_n - E_{n+m})t/\hbar} \right\rangle \int_0^t dt' \left\langle \sum_l c_{n,n+l} e^{i(E_n - E_{n+l})t'/\hbar} \right\rangle , \qquad (10)$$

This is exact for $t/t_H \to 0$, provided that $c_{n,n+l}$ does *not* rapidly decrease with $|l|$.[2)]

The first average in Eq. (10) reduces to $K(t)$ on using Eqs. (4) and (6). The second average in Eq. (10) differs from the return probability (8) by a remainder arising from the terms with $l > \mathcal{N}_0$. When $|E_n - E_{n+l}| > E_0$ the two wavefunctions are essentially uncorrelated, and $c_{n,n+l} = L^{-d}$. The remainder is therefore

$$\left\langle \sum_{l>\mathcal{N}_0} c_{n,n+l} e^{-i(E_n - E_{n+l})t/\hbar} \right\rangle = \left\langle \int_{E>E_0} \rho(E) e^{-iEt/\hbar} \right\rangle = 2\pi\hbar\rho\delta_0(t) ,$$

where $\delta_0(t)$ is a δ-like function which is zero for $t > t_0 = \hbar/E_0$. As t_0/t_H goes to zero in the thermodynamic limit considered, the integral in Eq. (10) is equal to $\pi\hbar\rho + \int_{0+}^t p(t')\,dt'$. Thefore, the factorisation (10) of the expression (9) leads to the relation (7) between $K(t)$ and $p(t)$ which is exact in the limit $t/t_H \to 0$.

Now we apply Eq. (7) to find the compressibility at the mobility edge. This regime is characterised by anomalous diffusion. This means that the diffusion coefficient D becomes time (and scale) dependent. In the absence of multifractality, it has been shown [15] by extending one-parameter scaling arguments to treat time- or frequency-dependent transport that $D(t) \propto t^{-1+2/d}$. Then (restoring dimensional factors) $p(t) \sim (Dt)^{-d/2} \sim \hbar\rho/t$. The numerator in Eq. (7) is therefore constant while the denominator is divergent in the universal limit, $L \to \infty$ with t/t_H fixed. Thus, in this case, $K(t) = 0$ for $t \ll t_H$ and the compressibility (5) would be zero at the mobility edge, as in the metal.

However, there is no *a priori* reason to extend one-parameter scaling in this way. In fact, wavefunction correlations at the mobility edge are characterised by

[2)] The conditions necessary to derive Eq. (10) are not satisfied in the insulator, where the dominant contiribution to the r.h.s. of Eq. (9) comes from the $l = 0$ term. Retaining only this term, Eqs. (9) and (8) yield correctly $K(t) = 1$ and hence $\chi = 1$.

a set of multifractal exponents [16,17]. The relevant fractal dimension for $p(t)$ is D_2 [18], Eq. (3), as it is expressed via $|\psi_n|^2 |\psi_m|^2$, according to Eq. (8). Multifractality reveals itself in the time-dependence of $p(t)$. Since for $l \ll E_0/\Delta$ the factor $c_{n,n+l} \sim |E_n - E_{n+l}|^{-\eta/d}$, we have [18]:

$$p(t) \sim V_0^{-\eta/d} (\hbar\rho/t)^{1-\eta/d}, \qquad \eta \equiv d - D_2 .$$

On subsituting this into Eq. (7), we note that the first, constant term in the denominator is negligible in the limit $L \to \infty$, with t/t_H fixed. Thus we obtain $K(t) = 2\eta/d$ for $t \ll t_H$.

Note that in this limit $R(s) = 0$ for $s \gg 1$, in agreement with the diagrammatic results of Ref. [7]. There are small t-dependent corrections to the leading behaviour of $K(t)$ that govern the large-s asymptotics of $R(s)$. These corrections, which will be discussed elsewhere, do not affect the limiting value $K(0)$: thus our main result, Eq. (2), follows immediately.

This result can be checked directly for $d = 2$. In this case, there is no Anderson transition but a crossover from the weak to strong localisation regime. It is known that, in the weak localisation regime, where the dimensionless conductance, $g_0 = 2\pi^2\rho D \gg 1$, the high moments of the inverse participation ratio exhibit behaviour characterstic of that for critical states in $d > 2$. For the orthogonal, unitary and symplectic ensembles labelled by $\beta = 1, 2$ or 4, the corresponding set of multifractal dimensions has been recently found non-perturbatively [19] to be $D_p = 2 - p/\beta g_0$, in agreement with the earlier renormalization group results [16,20]. This leads to $\eta = 2/\beta g_0$, and thus the limiting value of compressibility (2) is $\chi = 1/2\beta g_0$. On the other hand, the same value of compressibility has been obtained in this regime with the help of direct diagrammatic calcultations [9] (the corresponding Eq. (15) in Ref. [9] contains an extra factor of 2 due to explicit inclusion of spin degeneracy).

Another possibility [21] to check Eq.(2) is provided by a $1d$ system with random hopping integrals t_{ij} whose variance decreases as a power of the distance $|i - j|$: $\langle (t_{ij})^2 \rangle = A|i - j|^{2\alpha}$. This system is critical at $\alpha = 1$, and eigenstates are multifractal. Calculations similar to those described above for 2D systems show [21] that Eq. (2) is again exactly satisfied.

Finally, the relation (2) between the compressibility χ and the multifractality exponent can be checked with the help of independent numerical calculations of both these quantities. Let us discuss available numerical results. For the 3D Anderson model (which belongs to the orthogonal symmetry class), the multifractality exponent D_2 was found [22–25] to be $D_2 = 1.7 \pm 0.2$. Using Eq. (2) we have $\chi = 0.22 \pm 0.03$ which is in a good agreement with the numerical result $\chi = 0.2 \div 0.3$ [8,26–28]. The exponent η has been studied numerically also for the Anderson transitions in 2D systems belonging to the unitary (quantum-Hall regime) and the symplectic symmetry classes [24,29]. In both cases $\eta \approx 0.4$ turns out to be small. Thus we can predict that the level compessibility should be very small, $\chi \approx 0.1$, in the vicinity of these transitions. This could explain that in direct numerical simulations in the quantum-Hall regime [30] no linear contribution to Σ_2 has been found.

In summary, we have derived an explicit relationship between the compressibility of energy levels and the multifractal statistics of eigenfunctions in a disordered system near the Anderson transition. A non-zero compressibility implies that the one-parameter scaling is inapplicable to level statistics; for a hypothetical system where the one-parameter scaling description of the conductance could be extended to level correlations, no spectral compressibility would arise at the transition.

The relationship (2) holds exactly for 2D critical states and is in a reasonable agreement with existing numerical results.

We thank G. Montambaux, L. Schweitzer and I. K. Zharekeshev for helpful discussions on numerical results. Support from EPSRC grants GR/GO 2727 (J.T.C.) and GR/J35238 (I.V.L.) is gratefully acknowledged. Two of us (V.E.K. and I.V.L.) gratefully acknowledge the hospitality of the ITP in Santa Barbara extended to us at different stages of this work and partial support by the NSF under grant PHY94-07194.

---

1. L.P.Gor'kov and G.M.Eliashberg, Zh. Eksp. Teor. Fiz. **48**, 1407 (1965) [Sov. Phys. JETP **21**, 940 (1965)].
2. K.B.Efetov, Adv.Phys. **32**, 53 (1983).
3. B.L.Altshuler and B.I.Shklovskii, Zh. Eksp. Teor. Fiz. **91**, 220 (1986) [Sov. Phys. JETP **64**, 127 (1986)].
4. E.P.Wigner, Proc. Cambridge Philos. Soc. **47**, 790 (1951); F.J.Dyson, J. Math. Phys. **3**, 140 (1962).
5. M.L.Mehta, *Random matrices* Academic Press, Boston, 1991.
6. B.I.Shklovskii, B.Shapiro, B.R.Sears et al., Phys. Rev. B **47**, 11487 (1993).
7. V.E.Kravtsov, I.V.Lerner, B.L.Altshuler, and A.G.Aronov, Phys. Rev. Lett. **72**, 888 (1994); A.G.Aronov, V.E.Kravtsov, and I.V.Lerner, *ibid* **74**, 1174 (1995).
8. B.L.Altshuler, I.K.Zharekeshev, S.A.Kotochigova, and B.I.Shklovskii, Zh. Eksp. Teor. Fiz. **94**, 343 (1988) [Sov. Phys. JETP **67**, 625 (1988)].
9. V.E.Kravtsov and I.V.Lerner, Phys. Rev. Lett. **74**, 2563 (1995).
10. A.G.Aronov and A.D.Mirlin, Phys. Rev. B **51**, 6131 (1995).
11. J.T.Chalker, I.V.Lerner, and R.S.Smith, Phys. Rev. Lett., **77** 554 (1996).
12. J.T.Chalker, I.V.Lerner, and R.S. Smith, J. Math. Phys., in press (1996).
13. V.E.Kravtsov, in: Proc. of the 1996 Moriond Conf. (cond-mat/9603166).
14. F.J.Dyson, J. Math. Phys. **3**, 1191 (1962).
15. B.Shapiro and E.Abrahams, Phys. Rev. B **24**, 4889 (1981); Y.Imry, Y.Gefen and D.J.Bergman, Phys. Rev. B **26**, 3436 (1982).
16. F.Wegner, Z. Phys. B **36**, 209 (1980).
17. C.Castellani and L.Peliti, J. Phys. A **19**, L429 (1986).
18. J.T.Chalker and G.J.Daniell, Phys. Rev. Lett. **61**, 593 (1988); J.T.Chalker, Physica **A 167**, 253 (1990); B.Huckenstein and L.Schweitzer, Phys. Rev. Lett. **72**, 713 (1994).
19. K.B.Efetov and V.I.Falko, Europhys. Lett. **32**, 627 (1995).
20. B.L.Altshuler, V.E.Kravtsov, and I.V.Lerner, Zh. Eksp. Teor. Fiz. **91**, 2276 (1986) [Sov. Phys. JETP **64**, 1352 (1986)].
21. A.D.Mirlin, Y.V.Fyodorov, F.-M.Dittes et al., cond-mat/9604163, 1996.
22. C.M.Socoulis and E.N.Economou, Phys. Rev. Lett. **52**, 565 (1984).
23. M.Schreiber, Physica A **167**, 188, (1990);
24. T.Brandes, B.Huckestein and L.Schweitzer, cond-mat/9605062, 1996.
25. S.N.Evangelou, J. Phys. A **23**, L317 (1990).
26. I.Kh.Zharekeshev and B.Kramer, Jpn. J. Appl. Phys. , **34**, 4361 (1995).
27. D.Braun and G.Montambaux, Phys. Rev. B **52**, 13906 (1995).
28. B.R.Sears and H.B.Shore, unpublished (1994).
29. S.N.Evangelou, Phys. Rev. Lett. **75**, 2550 (1995).
30. M.Feingold, Y.Avishai, and R.Berkovits, Phys. Rev. B **52**, 8400 (1995).