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ON THE STATISTICAL MECHANICS OF ELESTIC MANIFOLDS WITH RANDOM INTERACTION

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We study the statistical mechanics of D-dimensional elastic manifolds, interacting via randomly distributed forces. It is shown, that this model can be mapped onto the statistical mechanics of disorder-induced roughening of a D-dimensional interface with D transverse degrees of freedom in a disordered medium. The roughness exponent ζ for the lateral deformations is calculated for different kinds of elastic response of the manifolds.

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The statistical mechanics and the driven dynamics of elastic manifolds in a disordered environment have been a subject of intensive theoretical investigations in the last ten years. Competition between elasticity and randomness gives rise to the existence of many metastable configurations of the system, which in turn results in the non-trivial scaling laws. Such models have been successfully used to explain the behaviour of a variety of physical systems, such as vortices in superconductors [1], interfaces (domain walls) in ordered media [2, 3], etc.

Recently some attempts were undertaken to apply these ideas and methods to a completely different kind of problems, namely, for description of the static and dynamic processes, accompanying dry friction between two solid bodies, and the role of elasticity in these processes. One can mention, for example, Refs.[4, 5], where different models for interacting disordered surfaces are studied, and also Ref.[6], where the universality properties are discussed. Although much efforts have been aimed towards explanation of the complex dynamic behaviour near the depinning transition, it is also interesting to study the equilibrium static properties of disordered surfaces in contact. The same question can be asked, for example, when studying the behaviour of "tethered surfaces" (i.e. the multidimensional generalizations of polymer chains) with quenched topological defects on polycrystalline or amorphous substrates [7].

The problem can be formulated as follows. Let us consider a flat D-dimensional elastic manifold with disorder, lying on a random rigid substate. We allow only for the deformations $\mathbf{u}(\mathbf{r})$ in all lateral directions, thus neglecting the "vertical" degree of freedom. The key question is how to take into account both the randomness of contacting surfaces and the interaction between them. It seems quite reasonable to incorporate both these features by assuming [6] that there are quenched scalar functions $\rho_1(\mathbf{r}_1)$ and $\rho_2(\mathbf{r}_2)$ on the manifold and the substrate respectively, which are independent Gaussian distributed random quantities with zero mean and the

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following correlation function:

$$\langle \rho_i(\mathbf{r}_1)\rho_j(\mathbf{r}_2)\rangle = \delta_{ij}K(\mathbf{r}_1 - \mathbf{r}_2), \tag{1}$$

where $K(R)$ is a short-ranged function, vanishing at $R > \sigma$. Then the interaction of the manifold with the substrate is ascribed to the local forces between these "quenched charges" ρ_i. The discretized mechanical version of such model is represented by a D-dimensional array of randomly charged blocks, connected by springs, on a rough substrate.

Position of r-th element of the manifold in the fixed reference frame is given by $\mathbf{R}(\mathbf{r}) = \mathbf{r} + \mathbf{u}(\mathbf{r})$. Hamiltonian of the system has the following form:

$$\mathcal{H} = H_{elas} + H_{int}, \tag{2}$$

where H_{elas} is the energy of elastic deformations. If the manifold is a membrane-like object, then H_{elas} has a standart form, that is quadratic in gradients of $\mathbf{u}$. However, if to consider the contacting surfaces of two solid bodies, then the bulk-mediated response dominates, i.e. the elastic interaction is non-local in real space [8] and non-analytical in the momentum representation. Both kinds of elasticity can therefore be described by the following hamiltonian:

$$H_{elas} = \frac{1}{2}\int \frac{d^D k}{(2\pi)^D}\ (\gamma_{\parallel} n_\alpha n_\beta + \gamma_\perp(\delta_{\alpha\beta} - n_\alpha n_\beta))|\mathbf{k}|^n u_\alpha(\mathbf{k})u_\beta(-\mathbf{k}), \tag{3}$$

where n=1 or 2, $\gamma_{\parallel}$ and $\gamma_\perp$ are the longitudinal and the transverse elastic moduli respectively, and $\mathbf{n} = \mathbf{k}/|\mathbf{k}|$.

The second term in (2) looks as follows:

$$H_{int} = \int d^D r\ \ U(\rho_1(\mathbf{r}), \rho_2(\mathbf{r} + \mathbf{u}(\mathbf{r}))), \tag{4}$$

where the potential energy U depends on the microscopic details of interaction. In what follows we restrict ourselves by consideration of the specific form of U, which seems to be the simplest reasonable assumption [6]:

$$U = V_0\rho_1\rho_2. \tag{5}$$

This expression corresponds to the randomly distributed spots of attraction or repulsion between the "quenched charges". Apart from being apparently the most suitable for analytical treatment, such electrostatic-like potential energy seems to provide a good model description for some real physical systems, for example, in the case of "tethered surfaces", where $\rho_1(\mathbf{r})$ can be thought as the density of defects; or for geometrically smooth contacting surfaces of two solid bodies in assumption, that there act some local adhesive forces between them, e.g., due to the presence of a surface chemical disorder.

If there were no interaction, then $\mathbf{u}(\mathbf{r}) = 0$. In the presence of interaction, the system tends to minimize its energy (at $T = 0$), and hence some non-zero lateral deformations arise. We are interested in finding the "roughness exponent" $\zeta(D, n)$ in the following expression:

$$\langle (u_\alpha(\mathbf{r}_1) - u_\beta(\mathbf{r}_2))^2\rangle \sim \delta_{\alpha\beta}|\mathbf{r}_1 - \mathbf{r}_2|^{2\zeta}, \tag{6}$$

where $\alpha = 1, .., D$. Angular brackets here denote average over the random variables ρ_1 and ρ_2. One should keep in mind that, although we borrow the terminology from the interface depinning problem, there is no "roughness" in our problem, so that ζ serves merely as a measure of the lateral deformations.

The equation, determining the equilibrium shape of the manifold, is

$$\frac{\delta H_{elas}}{\delta u_\alpha} = F_\alpha(\mathbf{r}, \mathbf{u}) \equiv \rho_1(\mathbf{r})\partial_\alpha \rho_2(\mathbf{r} + \mathbf{u}(\mathbf{r})). \tag{7}$$

In order to calculate ζ, one might proceed in the following manner: to expand the r.h.s. of Eq.(7) in powers of u, and then solve these equations perturbatively, step-by-step. In first approximation we have $F_\alpha(\mathbf{r}, \mathbf{u}) \approx \rho_1(\mathbf{r})\partial_\alpha \rho_2(\mathbf{r})$ – the random force regime (or the Larkin regime [9]). From (1) we then obtain:

$$\langle \mathbf{u}(\mathbf{k})\mathbf{u}(-\mathbf{k})\rangle \sim \frac{V_0^2 K_0^2}{\gamma^4 \sigma^{D+2}} \frac{1}{k^{2n}},$$

where $K_0 = \int d^D R \;\; K(R)$, and σ is the ultraviolet cutoff. Therefore,

$$\zeta_{\mathrm{rf}} = \frac{2n - D}{2}. \tag{8}$$

This means, that at $D \le D_c(n) = 2n$ the correlator of the lateral deformations diverges at large distances even in the absence of thermal fluctuations (disorder-induced roughening).

The random force expression (8) is valid provided the distance $L = |\mathbf{r}_1 - \mathbf{r}_2|$ does not exceed the Larkin lenght L_c [9], since at larger scales the perturbation theory fails due to the presence of many metastable minima of the energy (2). The following question arises: how to go beyond the random-force regime and calculate ζ at large distances?

In order to find the long-wavelenght behaviour of the system, we shall first demonstrate how our system can be mapped onto the statistical mechanics of a D-dimensional non-random interface with D transverse degrees of freedom, embedded in a $2D$-dimensional disordered medium, and then derive the functional renormalization group (FRG) equations for $K(\mathbf{R})$. For this purpose we evaluate the free energy, averaged over the quenched disorders ρ_1 and ρ_2, making use of the standart replica trick [10]:

$$\mathcal{F} = \langle \ln Z\rangle = \lim_{n\to 0} \frac{1}{n}(\langle Z^n\rangle - 1),$$

where

$$\langle Z^n\rangle = \left\langle \int \prod_{a=1}^{n} \mathcal{D}\mathbf{u}_a(\mathbf{r}) \exp\left(-\frac{1}{T}\sum_{a=1}^{n} \mathcal{H}[\mathbf{u}_a]\right)\right\rangle =$$

$$= \int \prod_a \mathcal{D}\mathbf{u}_a \exp\Big[-\frac{1}{T}\sum_a H_{elas}[\mathbf{u}_a] - \frac{1}{2}\mathrm{Sp}\ln\Big(\delta(\mathbf{R}_1 - \mathbf{R}_2) - $$

$$- \frac{V_0^2 K_0^2}{T^2}\sum_{a,b}\int d^D r \;\; \delta(\mathbf{R}_1 - \mathbf{r} - \mathbf{u}_a(\mathbf{r}))\delta(\mathbf{R}_2 - \mathbf{r} - \mathbf{u}_b(\mathbf{r}))\Big)\Big]$$

(here we substituted (1) with $K(\mathbf{R}) = K_0\delta(\mathbf{R})$, and denoted $\mathrm{Sp} = \int d^D R$). Logarithm in the r.h.s. of this expression can be expanded in powers of the interaction constant, resulting in

$$\langle Z^n \rangle = \int \prod_a \mathcal{D}\mathbf{u}_a \exp\Big[-\frac{1}{T}\sum_a H_{elas}[\mathbf{u}_a] + \\ + \frac{V_0^2 K_0^2}{2T^2} \sum_{a,b} \int d^D r \;\; \delta(\mathbf{u}_a(\mathbf{r}) - \mathbf{u}_b(\mathbf{r})) + \ldots \Big], \qquad (9)$$

where the ellipsis stands for the interaction terms of higher orders in V_0^2, containing four or more replica indices. For example:

$$H'^{(4)}_{\mathrm{int}} = \frac{1}{4}\left(\frac{V_0^2 K_0^2}{T^2}\right)^2 \sum_{a,b,c,d} \int d^D r_1 d^D r_2 \;\; \delta(\mathbf{r}_1 - \mathbf{r}_2 + \mathbf{u}_a(\mathbf{r}_1) - \mathbf{u}_b(\mathbf{r}_2)) \times \\ \times \delta(\mathbf{r}_1 - \mathbf{r}_2 + \mathbf{u}_c(\mathbf{r}_1) - \mathbf{u}_d(\mathbf{r}_2)).$$

But such terms are apparently irrelevant. Indeed, since $\zeta(D,n) < 1$, in the long-wavelenght limit the only contribution with $\mathbf{r}_1 = \mathbf{r}_2$ survives in r.h.s., i.e.

$$H^{(4)}_{\mathrm{int}} \sim a^D \left(\frac{V_0^2 K_0^2}{T^2}\right)^2 \sum_{a,b,c,d} \int d^D r \;\; \delta(\mathbf{u}_a(\mathbf{r}) - \mathbf{u}_b(\mathbf{r}))\delta(\mathbf{u}_c(\mathbf{r}) - \mathbf{u}_d(\mathbf{r})),$$

where a is some microscopic lenght scale. Power counting here yields the condition $4n - 3D - 2(D+4)\zeta < 0$, which is always fulfilled near $D_c = 2n$.

Expression (9) has exactly the same form (for $n = 2$) as the free energy of a D-dimensional elastic interface in a $2D$-dimensional disordered medium [11]. The lateral deformations $\mathbf{u}(\mathbf{r})$ substitute in our case the transverse degrees of freedom. Therefore our model of *two* interacting random manifolds falls into the universality class of *a single* interface in a medium with short-range correlated disorder. Although such relation has been explicitly established here for the specific (and most convenient for a quantitative analysis) form of U, we believe that this general conclusion does not actually depend on the microscopic details of interaction. Note also, that the same equivalence has been shown [6] to hold for the driven dynamics or our model near the depinning transition.

The second term in brackets in the r.h.s. of Eq.(9) can be expanded in power series:

$$K(\mathbf{u}_a - \mathbf{u}_b) = \sum_{m=1}^{\infty} \frac{K_{2m}}{(2m)!}(\mathbf{u}_a - \mathbf{u}_b)^{2m}.$$

Since under the scaling transformation $r \to \lambda r$ the temperature scales as $T \to \lambda^{-\Delta_T} T$, where $\Delta_T = n - D - 2\zeta$, and $\zeta = O(\epsilon)$, where $\epsilon = D_c - D$, we see, that the dimensionalities of all K_{2m} become positive at $D < D_c$: $\Delta_{2m} = D + 2\Delta_T + 2m\zeta \sim \epsilon > 0$. This means, that one has to renormalize *all* coefficients K_{2m}, or, in other words, we have to deal with the functional renormalization group [12]. In derivation of the functional flow equations we closely follow the procedure of Ref.[13]; one has only to take into account the non-standart form (3) of the elastic hamiltonian. The result looks as follows:

$$\frac{\partial K(\mathbf{R})}{\partial l} = (2n - D - 4\zeta)K(\mathbf{R}) + \zeta R_\alpha \partial_\alpha K(\mathbf{R}) +$$

$$+(1+\omega_1)\left(\frac{1}{2}\partial_\alpha\partial_\beta K(\mathbf{R})\partial_\alpha\partial_\beta K(\mathbf{R})-\partial_\alpha\partial_\beta K(\mathbf{R})\partial_\alpha\partial_\beta K(\mathbf{0})\right)+$$
$$+\omega_2\left(\frac{1}{2}\partial^2 K(\mathbf{R})\partial^2 K(\mathbf{R})-\partial^2 K(\mathbf{R})\partial^2 K(\mathbf{0})\right), \tag{10}$$

where

$$\omega_1=\frac{2\lambda(\lambda+D+2)}{D(D+2)}, \quad \omega_2=\frac{\lambda^2}{D(D+2)}, \quad \lambda=\frac{\gamma_\perp-\gamma_\parallel}{\gamma_\perp},$$

and e^{dl} is the current ratio of upper cutoffs. The important differences from the well-known FRG equations of the interface depinning problem [11, 13] arise in the first term in r.h.s., which stems from simple rescaling of the correlation function, and in the second-order terms, which recover the standart form, if $\gamma_\perp=\gamma_\parallel$.

The value of ζ can be estimated, using the simple Flory-type arguments, which amount to the requirement for both terms in brackets in Eq.(9) to scale in the same way [14]. Then we find in the case of long-ranged correlator $K(R)\sim R^{-\beta}$, that

$$\zeta_{\mathrm{F}}=\frac{2n-D}{4+\beta}. \tag{11}$$

However, as we know from the interface problem, this expression is valid only if β is small enough: $\beta<\beta_c$, when the correlator remains asymptotically unchanged under renormalization. At $\beta\geq\beta_c$ the simple arguments fail, since the renormalization of disorder becomes important, and $K(R)$ flows towards a new short-range fixed point [11].

In our case, the functional flow equation (10) contains an explicit dependence on the ratio of the elastic moduli, so that one might expect ζ to take a non-universal value as well. However, this conjecture turns out to be wrong, if, following the reasoning of Halpin–Healy [11] (which is supported by the scaling analysis in Ref.[15]), to assume that the roughness exponent is determined entirely by the asymptotic behaviour of the fixed-point solution of the FRG equations. One can easily obtain from Eq.(10), that at $R\to\infty$

$$K(R)\sim R^{-4-D+\frac{2n-D}{\zeta}}\exp\left(-\frac{\zeta R^2}{2(2n-D)}\right), \tag{12}$$

so that the short-ranged fixed point solution is *asymptotically universal* in the sence, that it does not depend on λ, and, moreover, it is *asymptoticaly exact*, because of the higher order terms in the r.h.s. of Eq.(10) would give only the exponentially small corrections to (12). When applying the arguments of Ref.[11], that the exponentially damped power-law function corresponds to the critical value of β, separating the long- and short-range fixed point regimes, we immediately obtain that $\beta_c=D/2$. After substitution in (11) we come to the final result:

$$\zeta(D,n)=\frac{2(2n-D)}{8+D}. \tag{13}$$

For the physically most interesting cases we obtain from (13) the following values of the roughness exponent: i) dry friction: a solid body on a disordered substrate: $\zeta(2,1)=0$ (the logarithmic divergencies); ii) two-dimensional elastic membrane on a disordered substrate: $\zeta(2,2)=2/5$; iii) one-dimensional dry friction: $\zeta(1,1)=2/9$; iv) two strings with random interaction: $\zeta(1,2)=2/3$ – this result is actually exact [16]. In all cases the roughness exponent due to the thermal fluctuations is less than ζ (zero-temperature fixed point).

It should be emphasized, that the conclusion about the roughness exponent to be universal and be given by a simple expression (13), has been obtained following a rather convincing and self-consistent approach of Refs.[11, 15], which establish the relation between ζ and the large distance behaviour of the renormalized disorder correlator (for a different point of view see Refs.[12, 13, 17]).

In conclusion, we studied a statistical model of random interaction between two elastic manifolds, which can be applied to various physical systems. This model is shown to fall into the universality class of a single non-random D-dimensional interface in a $D + D$-dimensional disordered medium. The roughness exponent for the lateral deformations is calculated, making use of the functional renormalization group technique.

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