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DYNAMICS OF FIELDS IN THE MODEL OF GAUGED NONLINEAR (2+1)D SCHRÖDINGER EQUATION

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We study the solutions of the equations of motion in the gauged (2+1)-dimensional nonlinear Schrödinger equation. The contribution of the Chern–Simons gauge fields leads to a significant decrease of the critical power of self-focusing. We also show that for appropriate boundary conditions in the model considered there exists a regime of turbulent motion of a hydrodynamic type.

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1. The nonlinear Schrödinger equation (NSE) is one of the basic models for nonlinear waves. The traditional field of application of the NSE has been nonlinear optics [1, 2], where it describes the propagation of wave beams in nonlinear dispersive media. The NSE also arises in the treatment of various nonlinear waves in hydrodynamics and plasma physics [3]. A most important area of application in this case is the problem of the detailed description of collapsing field distributions [4, 5]. With the opposite sign of the nonlinearity the NSE is used as the basic model [6, 7] of the low-dimensional field theory for describing vortices in the problem of Bose condensation.

Recent interest in problems involving the solution of the NSE in spatially 2D systems has arisen in connection with the special properties exhibited by (2+1)D systems when the NSE is furnished with a gauge field through the replacement of the ordinary derivatives by covariant ones. In the infrared limit the main contribution to the equation of motion of the gauge field in a (2+1)D system is given by the Chern–Simons (CS) term within the system under consideration. For a certain relation of the coupling constants the contribution of the gauge field to the Hamiltonian compensates the contribution from the nonlinearity. It leads to a soliton distribution of the field, which was found in Ref. [8]. The results of Ref. [8] have stimulated a number of papers [9, 10, 11] in this field.

The purpose of this paper to study the equation of motion in the gauged (2+1)D nonlinear Schrödinger (GNSE) model. The main focus of attention is an investigation of the structure of the collapsing distribution of the fields. Specifically, by means of numerical integration of the equation of motion we find the dependence of the critical power and of the effective width of the zero-energy mode on the coefficient k in front of the CS term.

2. We consider a system with Lagrangian density

$$\mathcal{L} = \frac{k}{2}\varepsilon^{\alpha\beta\gamma}A_\alpha\partial_\beta A_\gamma + i\Psi^*(\partial_t + iA_0)\Psi - \frac{1}{2}\left|(\nabla - i\mathbf{A})\Psi\right|^2 + \frac{g}{2}\left|\Psi\right|^4 . \tag{1}$$

The equations of motion have the form

$$i\partial_t\Psi = -\frac{1}{2}(\nabla - i\mathbf{A})^2\Psi + A_0\Psi - g|\Psi|^2\Psi , \tag{2}$$

$$[\nabla \times \mathbf{A}]_\perp = -\frac{1}{k}|\Psi|^2, \tag{3}$$

$$\partial_t A_i + \partial_i A_0 = -\frac{1}{k}\varepsilon_{ij} j_j . \tag{4}$$

Here g is the coupling constant and $\mathbf{j} = \mathrm{Im}\Psi^*(\nabla - i\mathbf{A})\Psi$ is the current density. The Hamiltonian for Eq. (1) is

$$H = \frac{1}{2}\int d^2\mathbf{r}\left(|(\nabla - i\mathbf{A})\Psi|^2 - g|\Psi|^4\right), \tag{5}$$

where the potential A_μ is expressed in terms of $|\Psi|^2$ in the following way:

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{k}\int d^2\mathbf{r}'\mathbf{G}(\mathbf{r}-\mathbf{r}')|\Psi|^2(\mathbf{r}',t), \tag{6}$$

$$A_0(\mathbf{r},t) = \frac{1}{k}\int d^2\mathbf{r}'\mathbf{G}(\mathbf{r}-\mathbf{r}')\mathbf{j}(\mathbf{r}',t). \tag{7}$$

The Green function $\mathbf{G}(\mathbf{r})$

$$G_i(\mathbf{r}) = \frac{1}{2\pi}\frac{\varepsilon_{ij}x_j}{r^2} \tag{8}$$

satisfies the equation

$$\nabla \times \mathbf{G}(\mathbf{r}) = -\delta^2(\mathbf{r}). \tag{9}$$

Since in the Hamiltonian formulation the potentials are uniquely determined by Eqs. (6), (7), the gauge freedom

$$A_\mu \longrightarrow A_\mu - \partial_\mu\varphi, \tag{10}$$

$$\Psi \longrightarrow e^{i\varphi}\,\Psi \tag{11}$$

is fixed. This is achieved by using the Coulomb gauge $\nabla\cdot\mathbf{A} = 0$ with the boundary conditions

$$\lim_{r\to\infty} r^2 A_i(\mathbf{r},t) = \frac{1}{2\pi k}\varepsilon_{ij}x_j N, \tag{12}$$

$$\lim_{r\to\infty} A_0(\mathbf{r},t) = 0. \tag{13}$$

The choice of the boundary condition (12) derives from the necessity of satisfying the Gauss' law (3) of CS dynamics: $\Phi = -\frac{1}{k}N$. Here the magnetic flux is Φ and the number of particles is $N = \int d^2\mathbf{r}|\Psi|^2$.

The equation of motion and the continuity equation, expressed in terms of the dimensionless fields $\rho \equiv \rho(x,y,t)$, $u \equiv u(x,y,t)$, $v \equiv v(x,y,t)$, $w \equiv w(x,y,t)$ and the coordinates obtained by the following substitutions $\Psi = |k|^{3/2}\rho e^{i\varphi}$, $A_0 = -\frac{k^2}{2}w - \partial_t\varphi$, $A_x = -k\,u + \partial_x\varphi$, $A_y = -k\,v + \partial_y\varphi$, $t \to \frac{-2}{k|k|}t$, $x \to \frac{x}{|k|}$, $y \to \frac{y}{|k|}$ have the form

$$\rho_{xx} + \rho_{yy} = -2C\rho^3 - \rho w + \rho(u^2 + v^2), \tag{14}$$

$$u_y - v_x = -\rho^2, \tag{15}$$

$$u_t - w_x = -2v\rho^2, \tag{16}$$

$$v_t - w_y = 2u\rho^2, \tag{17}$$

$$\rho^2_{\ t} = 2\left((u\rho^2)_x + (v\rho^2)_y\right) \tag{18}$$

with the parameter $C = g|k|$ and the notation $u_t = \partial_t u$, etc.

Let us assume that in the Coulomb gauge $\nabla \cdot \mathbf{A} = -u_x - v_y + \Delta\varphi = 0$ the phase φ satisfies the equation $\Delta\varphi=0$. Then the solution of the equation $u_x + v_y = 0$ may be expressed in terms of a function $a(x, y, t)$ in the following way: $u = a_y$, $v = -a_x$. In this case, Eqs. (15) and (18) have the form

$$a_{xx} + a_{yy} = -\rho^2 , \tag{19}$$

$$\rho^2_{\ t} + u\rho^2_{\ x} + v\rho^2_{\ y} = 0 . \tag{20}$$

The set of Eqs. (19) and (20) represents the "vorticity" form of the Navier–Stokes equations (Euler equations) for two-dimensional flows of ideal incompressible fluids, where the function $a(x, y, t)$ has the meaning of a stream function. Note that hydrodynamic analogies have been used previously for the solution of (1+1)D NSE problem [12, 17]. However, the present paper gives the first rigorous proof that the dynamics of the CS gauge field in the framework of the GNSE model (in the particular case of the Coulomb gauge with $\Delta\varphi = 0$) is equivalent to the two-dimensional equations of motion of ideal incompressible fluid.

Let us consider for example the case when the ansatz for the field $\Psi(x, y, t)$ corresponds to the generalized lens transformation [10]

$$\Psi(\mathbf{r}, t) = \frac{\Phi(\boldsymbol{\zeta}, \tau)}{g(\tau)} \exp\left(-ib(\tau)\boldsymbol{\zeta}^2/2 + i\lambda\tau\right) . \tag{21}$$

Here $\boldsymbol{\zeta} = \mathbf{r}/g(\tau)$, $\tau = \int_0^t du\,[f(u)]^{-2}$, and $b(\tau) = -f_t f = -g_\tau g$. The gauge potential transforms [8] upon such a substitution as follows:

$$\mathbf{A}(\mathbf{r}, t) \to [g(\tau)]^{-1}\,\mathbf{A}(\boldsymbol{\zeta}, \tau) , \tag{22}$$

$$A_0(\mathbf{r}, t) \to [g(\tau)]^{-2}\left[A_0(\boldsymbol{\zeta}, \tau) - b(\tau)\boldsymbol{\zeta}\mathbf{A}(\boldsymbol{\zeta}, \tau)\right] , \tag{23}$$

while relations (6) and (7) are preserved, where the function $\rho = |\Phi|$. After these transformations Eq. (2) becomes

$$i\partial_\tau\Phi + (\beta\boldsymbol{\zeta}^2 - \lambda)\Phi = -\frac{1}{2}(\nabla - i\mathbf{A})^2\Phi + A_0\Phi - g|\Phi|^2\Phi , \tag{24}$$

because the function $\beta(\tau) = (b^2 + b_\tau)/2 = -f^3 f_{tt}/2$ does not equal zero in the case $\varphi(x, y, t) \sim b(x^2 + y^2)$, $b(t) \not= t_0 - t$. However if we are interested in collapsing solutions with [14, 10] $f^2(t) \sim (t_0 - t)/\ln[\ln(t_0 - t)]$, the structure of the self-similar nonlinear core [10] of the solution is described by the solutions of the equation

$$-\lambda\Phi = -\frac{1}{2}(\nabla - i\mathbf{A})^2\Phi + A_0\Phi - g|\Phi|^2\Phi . \tag{25}$$

3. For the numerical analysis of the solutions of Eq. (25) we use the method of the stabilizing multiplier [15]. The iteration approach for Eq. (25) has the form

$$\Phi_{n+1} = M_n F^{-1}\left(G(p)F\left(-2C\Phi_n^3 + j\Phi_n(u^2 + v^2 - w)_n\right)\right) , \tag{26}$$

$$M_n = \left(\frac{\int d^2p (F\Phi_n)^2}{\int d^2p G(p) F\Phi_n F(-2C\Phi_n^3 + j\Phi_n(u^2+v^2-w)_n)} \right)^{\alpha} . \quad (27)$$

Here F (F^{-1}) are the operators of the direct (inverse) Fourier transform, $G(p) = -\left(p^2+\lambda\right)^{-1}$. The multiplier is $j=1$ or $j=0$, respectively, depending on whether the nonlinear contribution of the gauge field in Eq. (26) is taken into account or neglected. In the case $j=0$ the usual normalization in the NSE corresponds to $C=1/2$. Without loss of generality we shall suppose below that $\lambda=1$. In the simulation carried out in the present study we have used the value $\alpha=3/2$, which rapidly gives the value $M_n=1$ for the stabilizing multiplier.

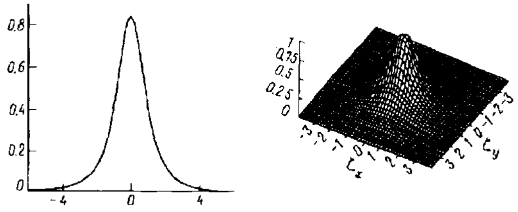

Fig.1. Plot of the function $\rho(\zeta_x,0)=\rho(0,\zeta_y)$ and the surface $\rho(\zeta_x,\zeta_y)$.

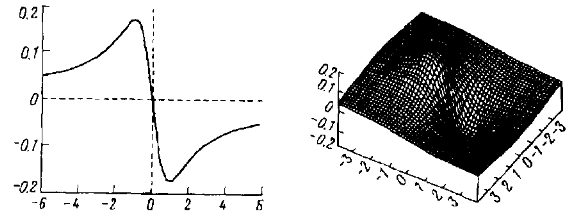

Fig.2. Plot of the function $u(\zeta_x,0)$ and the surface $u(\zeta_x,\zeta_y)$

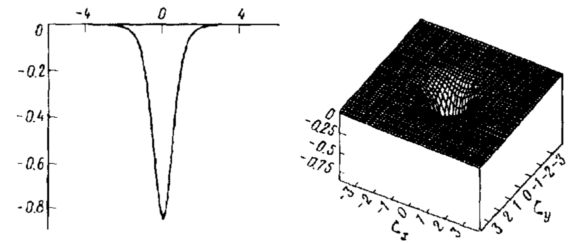

Fig.3. Plot of $w(\zeta_x,0)=w(0,\zeta_y)$ and the surface $w(\zeta_x,\zeta_y)$

Figures 1–3 show the configurations of the fields ρ, u and w for the specific value of the parameter $C=4$. Using the function ρ obtained, we computed the dependence of the critical power N and of the effective width $\langle R^2\rangle = N^{-1}\int d^2\zeta\, \boldsymbol{\zeta}^2 \rho^2(\boldsymbol{\zeta})$ on the parameter C. The results of calculations are given in Table.

j	C	N	$\langle R^2\rangle$
0	0.5	11.703	1.2607
1	2.85	3.6483	1.2384
1	3	2.9216	1.2464
1	5	1.2825	1.2579
1	10	0.5973	1.2600
1	100	$5.8528\cdot10^{-2}$	1.26066

4. It is seen from Eqs. (14)-(18) that if we neglect the CS gauge fields ($j=0$ in Eq. (26)) the dependence of the particle number N on the parameter $C=g|k|$ has the form $N=N_0/C$. This dependence is shown by the dotted line in Fig. 4. The contribution of the CS gauge fields ($j=1$ in Table 1) leads to a sharp decrease in the values of N. The effective width $\langle R^2 \rangle$ changes slightly. As expected, for a fixed value of the parameter C in the region $C \geq 3$ the number $N_{j=1}(C)$ is always greater than $N_{j=0}(C)$, because the CS gauge fields describe an effective repulsion.

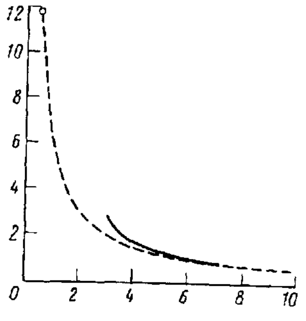

Fig.4. Number of particles N as a function of the parameter C without taking into account the gauge (dotted line) and with the gauge field (solid line). The point denotes the value $N(0.5)=11.703$

The present results correspond to the structure of the nonlinear core of the solution obtained by the lens transformation for the special value $\beta=0$ of the function $\beta(\tau)$ when $b(\tau)=1/(\tau_0+\tau)$. In this case the generalized lens transformation coincides with the conformal symmetry transformation [8] of the model. That is the reason why the form of the equation of motion (14) of the full model after the lens transformation coincides with Eq. (25). It will be very useful to compare the results obtained by the lens transformation for a finite function $\beta(\tau)$ in Eq. (24) with the results of a simulation using the full equations of motion (14)-(18) in the collapse regime.

Strong Langmuir turbulence in plasmas is usually described by the solutions of the NSE (Eq. (21)). It is assumed that a cascade of randomly distributed self-similar collapsing fields is generated. In this paper we show that the specific features of spatially two-dimensional systems may lead to the traditional picture of turbulence associated with the Euler equations. However, for the hydrodynamic mechanism of turbulence (HMT) to be involved, it is necessary that a linear profile of the phase $\varphi(x,y)$ exists in each mode. This implies that the nonlinear (in x and y) contributions to the temporal evolution of the phase are small. One of the media in which the HMT can play a role is an optical medium with random inhomogeneous guiding surfaces. Reflecting from the surfaces, the wave fronts acquire random directions of propagation. For media with weak Kerr nonlinearity, the nonlinear phase disturbance from adjacent points will not be important.

If the phase of the field $\Psi(x,y,t)$ describes the longitudinal part in the gauge potential *completely*, evolution of field configurations is determined *only by temporal dependence of the gauge field*. We show that in this case the equations for the gauge field coincide with the equations of motion of an ideal fluid. The effects of the manifestation of gauge field in classical systems with nontrivial topology, including the swimming motion at low Reynolds number within the $2+1D$

hydrodynamics are well known [16]. A new feature is the fact that the basis for the $2D$ turbulence in this case is chaotic dynamics of the CS gauge field.

The CS action with appropriate boundary conditions is a way to classify conformal field theories [13]. The tools of the conformal field theory, in its turn, may be used [18] to study the $2D$ turbulence. We show that within the model under consideration the connection between the dynamics of CS fields and $2D$ turbulence problem may be stated beyond the application of the conformal field theory. This dependence can be represented considering the evolution of closed current lines. Stochastization near the contour link points within formulation of $2D$ hydrodynamics of an ideal fluid in terms of contour variables [19] was discovered in Ref. [20].

In conclusion, we have studied numerically the structure of the collapsing mode in the GNSE model, observed a strong reduction of the critical power N in spatially two-dimensional systems as compared to the conventional values, and shown that in the case of general boundary conditions the phenomenon of collapse inhibits the development of turbulence according to the hydrodynamic scenario.

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1. V.I.Talanov, Sov. Radiophys. **9**, 260 (1966).
2. S.N.Vlasov, V.A.Petrishchev, and V.I.Talanov, Radiophys. and Quant. Electron. **14**, 1062 (1974).
3. See, for example, A.G. Litvak, *Dynamic nonlinear electromagnetic phenomena in plasma*, In Review of Plasma Physics, Ed. M.A.Leontovich, Consultants Bureau, N.Y. 1986, **10**.
4. A.G.Litvak, V.A.Mironov, and A.M.Sergeev, Phys. Scr. T **30**, 57 (1990).
5. V.E.Zhakharov, Sov. Phys. JETP **35**, 908 (1972).
6. V.L.Ginzburg and L.P.Pitaevski, Sov Phys JETP **7**, 858 (1958).
7. E.A.Kuznetsov and S.K.Turitsyn, Sov. Phys. JETP **55**, 844 (1982); **67**, 1583 (1988).
8. R.Jackiw and S.Y.Pi, Phys. Rev. Lett. **64**, 2969 (1990); Prog. Theor. Phys. Suppl. **107**, 1 (1992).
9. I.V.Barashenkov and A.O.Harin, Phys. Rev. Lett. **72**, 1575 (1994); Phys. Rev. D **52**, 2471 (1995).
10. L.Bergé, A. de Bouard, and J.C.Saut, Phys. Rev. Lett. **74**, 3907 (1995).
11. M.Knecht, R.Pasquier, and J.Y.Pasquier, J. Math. Phys. **36**, 4181 (1995).
12. A.Schwartzburg, In *Nonlinear Electromagnetics*, Ed. P.L.E. Uslenghi, Academic Press, N.Y., 1980.
13. G.Moore and N.Seiberg, Phys. Lett. B **212**, 451 (1988); **220**, 422 (1989).
14. G.M.Fraiman, Sov. Phys. JETP **61**, 228 (1985).
15. V.I.Petviashvili, Plasma Phys. **2**, 469 (1976).
16. A.Shapere and F.Wilczek, J. Fluid Mech. **198**, 557 (1989).
17. C.Nore, M.Abid, and M.Brachet, Lect. Note in Phys. **462**, 103 (1995).
18. A.M.Polyakov, Nucl. Phys. **B 396**, 367 (1993).
19. N.J.Zabusky, M.H.Hughes, and K.V.Roberts, J. Comp. Phys. **30**, 96 (1976).
20. A.A.Migdal, In *Nonlinear Waves. Patterns and Bifurcations*, Eds. A.V.Gaponov-Grekhov and M.I.Rabinovich, Nauka, Moscow, 1987; M.E.Aginstein and A.A. Migdal, In *Cybernetics Problems*, Eds. R.E.Sagdeev and A.A.Migdal, Sci. Council for Cybernetics Problems, Moscow, **107**, 114 (1987).