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ON THE QUANTIZATION OF $SU(3)$ SKYRMIONS[1)]

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The quantization condition derived previously for $SU(2)$ solitons quantized with $SU(3)$ collective coordinates is generalized for $SU(3)$ skyrmions with strangeness content different from zero. Quantization of the dipole-type configuration with large strangeness content found recently is considered as an example.

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1. The chiral soliton approach [1] allows not only to describe the properties of baryons with a rather good accuracy [2], [3] but also to make some predictions for the spectrum of states with baryon number $B > 1$ [4]-[6]. The quantization of the bound states of skyrmions, their zero modes first of all, is a quite necessary step towards realization of this approach. Different aspects of this problem have been considered beginning with the papers [2], [7] and [4], however, the complete treatment allowing the consideration of arbitrary $SU(3)$ skyrmions was absent till now.

In the sector with $B=2$ besides the $SO(3)$ hedgehog with the lowest quantum state being interpreted as H-dibaryon [4] the $SU(2)$ torus - bound $B=2$ state - was discovered almost 10 years ago [8]. The position of the known $B=2$ classical configurations representing local minima in $SU(3)$ configuration space is shown on Fig.1 in the plane with scalar strangeness content SC as Y axis and the difference of U- and D-contents as X-axis. Since the sum of all scalar contents equals to 1 they are defined uniquely in each point of this plot. The $SO(3)$ hedgehog (1) has all contents equal to 1/3. There are 3 torii in 3 different $SU(2)$ subgroups of $SU(3)$, the $u-d$ symmetric state (2) with $SC=0$ being of special interest. The dipole type state *5* found recently [9] has the binding energy about half of that of the torus.

The quantization of zero modes of solitons has been done previously in few different cases: for $SU(2)$ solitons rotated in $SU(2)$ and $SU(3)$ configuration spaces of collective coordinates [2], [7], [4], and also for $SO(3)$ solitons [4]. In the first case the quantization condition known as Guadagnini's one [7] was established, see also [10]. The quantization of $SU(2)$ torus leads to predictions of rich spectrum of strange dibaryons [11]. However, these kinds of solitons are only particular cases, the quantization procedure of arbitrary $SU(3)$ solitons should be developed. This is a subject of present paper.

2. Let us consider the Wess–Zumino (WZ) term in the action which defines the quantum numbers of the system in the quantization procedure. As usually, we introduce the time-dependent collective coordinates for the quantization of zero

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modes according to the relation: $U(\mathbf{r},t) = A(t)U_0(\mathbf{r})A^\dagger(t)$. The integration by parts is possible then in the expression for the WZ-term in the action [12]

$$S^{WZ} = \frac{-iN_c}{240\pi^2}\epsilon_{\mu\nu\alpha\beta\gamma}\int_\Omega Tr\bar{L}_\mu\bar{L}_\nu\bar{L}_\alpha\bar{L}_\beta\bar{L}_\gamma d^5x', \tag{1}$$

Ω being the 5-dimensional region with the 4-dimensional space-time as its boundary. We obtain then for the WZ-term contribution to the lagrangian of the system:

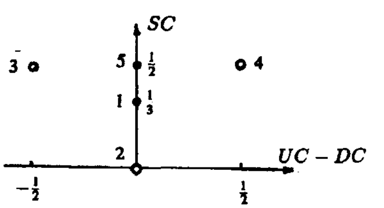

Fig.1. The position of different classical configurations with $B = 2$ in the plane $(UC - DC)$, SC. UC, DC and SC are scalar quark contents of the soliton. $UC = (1 - ReU_{11})/(3 - ReU_{11} - ReU_{22} - ReU_{33})$, etc. U_{ii} are the diagonal matrix elements of the unitary matrix U. (1) is the $SO(3)$ hedgehog, (2),(3) and (4) are $SU(2)$ torii in (u,d), (d,s) and (u,s) subgroups of $SU(3)$, (5) is the dipole-type configuration found recently

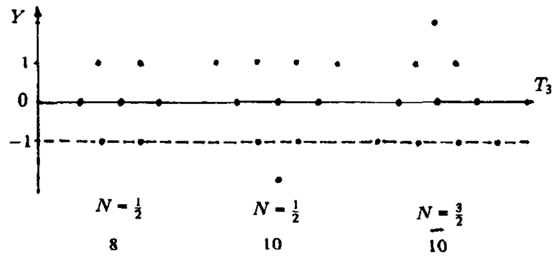

Fig.2. $T_3 - Y$ -diagrams for the lowest $SU(3)$ multiplets allowed for the case of $SU(2)^3$ configurations: octet $(1,1)$, decuplet $(3,0)$ and antidecuplet $(0,3)$. Dashed line indicates isomultiplets with $Y = Y^{min} = -1$, $T = N$

$$L^{WZ} = \frac{-iN_c}{48\pi^2}\epsilon_{\alpha\beta\gamma}\int TrA^\dagger\dot{A}(U_0L_\alpha L_\beta L_\gamma U_0^\dagger + L_\alpha L_\beta L_\gamma)d^3x \tag{2}$$

where $\bar{L}_\mu = U^\dagger d_\mu U$, $L_\alpha = U_0^\dagger d_\alpha U_0 = iL_{k,\alpha}\lambda_k$, or

$$L^{WZ} = \frac{N_c}{24\pi^2}\int\sum_{k=1}^{k=8}\omega_k WZ_k d^3x = \sum_{k=1}^{k=8}\omega_k L_k^{WZ} \tag{3}$$

with angular velocities of rotation in the configuration space defined in usual way,

$$A^\dagger\dot{A} = -\frac{i}{2}\omega_k\lambda_k.$$

Summation over repeated indices is assumed here and further. Functions WZ_k can be expressed through the Cartan-Maurer currents $L_{k,i}$:

$$WZ_i = (R_{ik}(U_0) + \delta_{ik})\widetilde{WZ}_k, \tag{4}$$

where

$$\widetilde{WZ}_1 = -(L_1, L_4L_5 + L_6L_7) - (L_2L_3L_8)/\sqrt{3} - 2(L_8, L_4L_7 - L_5L_6)/\sqrt{3},$$

$$\widetilde{WZ}_2 = -(L_2, L_4L_5 + L_6L_7) - (L_3L_1L_8)/\sqrt{3} - 2(L_8, L_4L_6 + L_5L_7)/\sqrt{3},$$

$$\widetilde{WZ}_3 = -(L_3, L_4L_5 + L_6L_7) - (L_1L_2L_8)/\sqrt{3} - 2(L_8, L_4L_5 - L_6L_7)/\sqrt{3},$$

$$\widetilde{WZ}_4 = -(L_4, L_1L_2 - L_6L_7) + (L_3, L_2L_6 + L_1L_7) - (L_8, L_1L_7 + L_2L_6 + L_3L_5)/\sqrt{3},$$

$$\widetilde{WZ}_5 = -(L_5, L_1L_2 - L_6L_7) - (L_3, L_1L_6 - L_2L_7) - (L_8, L_2L_7 - L_1L_6 - L_3L_4)/\sqrt{3},$$

$$\widetilde{WZ}_6 = (L_6, L_1L_2 + L_4L_5) - (L_3, L_1L_5 - L_2L_4) - (L_8, L_1L_5 - L_2L_4 - L_3L_7)/\sqrt{3},$$

$$\widetilde{WZ}_7 = (L_4, L_1L_2 + L_4L_5) + (L_3, L_2L_5 + L_1L_4) - (L_8, L_3L_6 - L_1L_4 - L_2L_5)/\sqrt{3},$$

$$\widetilde{WZ}_8 = -\sqrt{3}(L_1L_2L_3) + (L_8L_4L_5) + (L_8L_6L_7). \tag{5}$$

$(\mathbf{L}_1\mathbf{L}_2\mathbf{L}_3)$ denotes the mixed product of vectors $\mathbf{L}_1$, $\mathbf{L}_2$, $\mathbf{L}_3$. The real orthogonal matrix $R_{ik}(U_0) = 1/2\mathrm{Tr}\lambda_i U_0 \lambda_k U_0^\dagger$.

It should be noted that the results of calculation according to (5) depend on the orientation of the soliton in the $SU(3)$ configuration space.

When solitons are located in the (u,d) $SU(2)$ subgroup of $SU(3)$ only L_1, L_2 and L_3 are different from zero. WZ and $\tilde{WZ}$ are both proportional to the B-number density and the well known quantization condition by Guadagnini [7] rederived in [10] takes place,

$$Y_R = \frac{2}{\sqrt{3}} dL^{WZ}/d\omega_8 = N_c B/3 \tag{6}$$

where Y_R is the so called right hypercharge characterizing the $SU(3)$ irrep under consideration. This relation will be generalized here to

$$Y_R^{min} = \frac{2}{\sqrt{3}} dL^{WZ}/d\omega_8 = N_c B(1 - 3SC)/3. \tag{7}$$

This formula was checked for several cases.

a) We can rotate any $SU(2)$ soliton by arbitrary constant $SU(3)$ matrix containing $U_4 = exp(-i\nu\lambda_4)$. In this case $SC = \frac{1}{2}sin^2\nu$, both WZ_8, $\widetilde{WZ}_8$ are proportional to $R_{88} = 1 - \frac{3}{2}sin^2\nu$. As a result, the relation (7) is fulfilled exactly. Solitons (3) and (4) on Fig.1 can be obtained from (u,d) soliton (2) by means of U_4 or U_2U_4 rotations and satisfy relation (7).

b) For the $SO(3)$ hedgehog $SC = 1/3$, [6] and $L^{WZ} = 0$, [4] which satisfies (7) again.

c) We obtained (7) numerically for the solitons of the type $U = U_L(u,s)U(u,d)U_R(d,s)$ [9] with $U(u,d) = \exp(ia\lambda_2)\exp(ib\lambda_3)$ and $U_L(u,s)$ and $U_R(d,s)$ being deformed interacting $B = 1$ $SU(2)$ hedgehogs. For this ansatz we had for rotated $SU(3)$ Cartan–Maurer (CM) currents [9]:

$$\begin{aligned}
&\tilde{L}_{1i} = s_a c_a l_{3i}, && \tilde{L}_{2i} = d_i a, \\
&\tilde{L}_{3i} = (c_{2a} l_{3i} - r_{3i})/2 + d_i b, && \tilde{L}_{4i} = l_{1i} c_a, \\
&\tilde{L}_{5i} = c_a l_{2i}, && \tilde{L}_{6i} = l_{1i} s_a + r_{1i}(b), \\
&\tilde{L}_{7i} = s_a l_{2i} + r_{2i}(b), && \tilde{L}_{8i} = \sqrt{3}(l_{3i} + r_{3i})/2.
\end{aligned} \tag{8}$$

$\tilde{L} = TLT^{\dagger}$, $U_0 = VT$, $V = U(u,s)exp(ia\lambda_2)$, $T = exp(ib\lambda_3)U(d,s)$. $s_a = sina$, $c_a = cosa$, etc., in terms of $SU(2)$ C-M currents $l_{k,i}$ and $r_{k,i}$ $(i,k = 1,2,3)$ and functions a and b. In this case only the integral over the function $\widetilde{WZ}_8$ is different from zero ($N_c = 3$):

$$\frac{1}{2\sqrt{3}\pi^2}\int \widetilde{WZ}_8 d^3x = \frac{1}{4\pi^2}\int [(\mathbf{l}_1\mathbf{l}_2\mathbf{l}_3) + (\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3)]d^3x = -(B_L + B_R)/2 \qquad (9)$$

where B_L and B_R are the baryon numbers located in left (u,s) and right (d,s) $SU(2)$ subgroups of $SU(3)$. We should calculate (3), (7) with $WZ_8 = (R_{8k}(V) + R_{k8}(T))\widetilde{WZ}_k$. The contribution $-(B_L + B_R)/2$ also appears, together with some additional terms which turned out to be very small numerically. We obtained $SC = 0.49$ and $Y_R^{min} = -0.96$.

It is natural to suggest that (7) holds for any $SU(3)$ skyrmions.

3. The expression for the rotation energy of the system depending on the angular velocities of rotations in $SU(3)$ collective coordinates space can be written in such a form:

$$L_{rot} = \frac{F_\pi{}^2}{16}(\tilde{\omega}_1^2 + \tilde{\omega}_2^2 ... + \tilde{\omega}_8^2)$$

$$+\frac{1}{8e^2}\Bigg\{\mathbf{s}_{12}^2 + \mathbf{s}_{23}^2 + \mathbf{s}_{31}^2 + \mathbf{s}_{45}^2 + \mathbf{s}_{67}^2 + \frac{3}{4}\left(\mathbf{s}_{48}^2 + \mathbf{s}_{58}^2 + \mathbf{s}_{68}^2 + \mathbf{s}_{78}^2\right) +$$

$$+\frac{1}{4}\left(\mathbf{s}_{46}^2+\mathbf{s}_{47}^2+\mathbf{s}_{56}^2+\mathbf{s}_{57}^2+\mathbf{s}_{14}^2+\mathbf{s}_{15}^2+\mathbf{s}_{16}^2+\mathbf{s}_{17}^2+\mathbf{s}_{24}^2+\mathbf{s}_{25}^2+\mathbf{s}_{26}^2+\mathbf{s}_{27}^2+\mathbf{s}_{34}^2+\mathbf{s}_{35}^2+\mathbf{s}_{36}^2+\mathbf{s}_{37}^2\right) +$$

$$+\frac{\sqrt{3}}{2}\Big(\mathbf{s}_{84}(\mathbf{s}_{16}+\mathbf{s}_{34}-\mathbf{s}_{27})+\mathbf{s}_{85}(\mathbf{s}_{17}+\mathbf{s}_{26}+\mathbf{s}_{35})+\mathbf{s}_{86}(\mathbf{s}_{14}+\mathbf{s}_{25}-\mathbf{s}_{36})+\mathbf{s}_{87}(\mathbf{s}_{15}-\mathbf{s}_{24}-\mathbf{s}_{37})\Big) +$$

$$+\frac{3}{2}\Big(\mathbf{s}_{12}(\mathbf{s}_{45}+\mathbf{s}_{76}) + \mathbf{s}_{23}(\mathbf{s}_{47}+\mathbf{s}_{65}) + \mathbf{s}_{13}(\mathbf{s}_{64}+\mathbf{s}_{75}) + \mathbf{s}_{45}\mathbf{s}_{67}\Big)\Bigg\} \qquad (10)$$

Here $\mathbf{s}_{ik} = \tilde{\omega}_i\mathbf{L}_k - \tilde{\omega}_k\mathbf{L}_i$, $i,k = 1,2...8$ are the $SU(3)$ indices. The expression for static energy can be obtained from (10) by means of substitution $\mathbf{s}_{ik} = 2[\mathbf{L}_i\mathbf{L}_k]$, [9]. The functions $\tilde{\omega}_i$ are connected with the body fixed angular velocities of $SU(3)$ rotations by means of transformation (see (8) above):

$$\tilde{\omega} = V^{\dagger}\omega V - T\omega T^{\dagger}, \ \text{or} \quad \tilde{\omega}_i = (R_{ik}(V^{\dagger}) - R_{ik}(T))\omega_k = R_{ik}\omega_k \qquad (11)$$

$R_{ik}(V^{\dagger}) = R_{ki}(V)$ and $R_{ik}(T)$ are real orthogonal matrices, $i,k = 1,...,8$. For example,

$$R_{81} = -\frac{\sqrt{3}}{2}s_{2a}(f_1^2 + f_2^2), \quad R_{82} = 0, \quad R_{88} = \frac{3}{2}(q_1^2 + q_2^2 - f_1^2 - f_2^2), \text{etc}, \qquad (12)$$

$\tilde{U}(u,s) = f_0 + i\bar{\tau}_k f_k$, $\tilde{U}(d,s) = q_0 + i\tilde{\tau}_k q_k$, $k = 1,2,3$, $\bar{\tau}$ and $\tilde{\tau}$ are the Pauli matrices corresponding to (u,s) and (d,s) $SU(2)$ subgroups.

8 diagonal moments of inertia and 28 off-diagonal define the rotation energy – quadratic form in $\omega_i\omega_k$ – according to (10), (11). In the case of strange skyrmion molecule we obtained 4 different diagonal moments of inertia: $\Theta_1 = \Theta_2 = \Theta_N$; Θ_3; $\Theta_4 = \Theta_5 = \Theta_6 = \Theta_7 = \Theta_S$ and Θ_8. Numerically the difference between Θ_N and Θ_3 is small and both are about twice smaller than Θ_S. Θ_8 is a bit greater than Θ_S (see

Table[2)]). In view of symmetry properties of the configuration many off-diagonal moments of inertia are equal to zero. Few of them are different from zero, but at least one order of magnitude smaller than diagonal inertia: Θ_{38}, Θ_{46}, Θ_{57}. By this reason we shall neglect them here for the estimates. The hamiltonian of the system can be obtained by canonical quantization procedure [2],[7],[4] (we take the angular momentum $J = 0$) in such simplified form:

$$E_{rot} = \frac{C_2(SU_3) - 3Y_R^2/4}{2\Theta_S} + \frac{N(N+1)}{2}\left(\frac{1}{\Theta_N} - \frac{1}{\Theta_S}\right) + \frac{3(Y_R - Y_R^{min})^2}{8\Theta_8} \tag{13}$$

$C_2(SU_3) = \frac{1}{3}(p^2 + q^2 + pq) + p + q$, N is the right isospin (see Fig.2), p, q are the numbers of the upper and low indices in the tensor describing the $SU(3)$ irrep (p, q).

	B	M	Θ_N	Θ_S	Θ_3	Θ_8	Θ_{38}
FS	1	1702	3.95	1.62	–	–	–
FS	2	3334	2.91	5.19	1.99	5.71	0.47
FSB	1	1982	1.98	0.73	–	–	–
FSB	2	3900	1.44	2.69	0.99	2.85	0.19

It is clear from this expression that for $\Theta_8 \to 0$ the right hypercharge $Y_R = Y_R^{min} = \frac{2}{\sqrt{3}} L_8^{WZ}$, otherwise the quantum correction due to ω_8 will be infinite. For solitons located in (u, d) $SU(2)$ $\Theta_8 = 0$ and $Y_R = \frac{2}{\sqrt{3}} L_8^{WZ} = B$ - the quantization condition [7],[10] with $N_c = 3$.

For the skyrmion molecule found in [9] $L_8^{WZ} \approx -\frac{\sqrt{3}}{2}$, or $Y_R^{min} \approx -1$, as it was explained above. The last term in (13) is absent for $Y_R = -1$, and because of the evident constraints

$$(p + 2q)/3 \geq Y_R \geq -(q + 2p)/3 \tag{14}$$

the following lowest $SU(3)$ multiplets are possible: octet, $(p, q) = (1, 1)$, decuplet $(3, 0)$ and antidecuplet $(0, 3)$, Fig.2. The sum of the classical mass of the soliton and rotational energy for the $B = 2$ octet, 10 and $\overline{10}$ is equal to ~ 4.44, 5.0 and 5.5 *Gev* for $Y_R = -1$. The octets with $Y_R = 0$ and 1 have masses 4.9 and 5.0 *Gev*. This should be compared with central values of masses of $B = 1$ octet and decuplet 2.64 and 3.05*Gev* [3]. The absolute values of the masses of both $B = 1$ and 2 states are controlled by the Casimir energies which make contribution of N_c^0 into the masses of configurations [13]-[16]. However, the dipole-type configuration does not differ much from the $B = 2$ configuration in the product ansatz which we used as a starting one in our calculations [9]. By this reason the Casimir energy of the $B = 2$ dipole should be close to twice of that for $B = 1$ soliton, and will be canceled in the binding energies of dibaryons. We can conclude therefore that most of the $B = 2$ octet and decuplet states should be bound. The nonstrange state appears for the first time within the antidecuplet and is unbound.

The mass splittings inside multiplets are defined as usually by flavor symmetry breaking (FSB) terms in the lagrangian. In this case, since we start from the soliton with $SC \approx 0.5$, the FSB terms are squeezed by a factor about ~ 3 due to the smaller dimensions of the kaon cloud in comparison with the pion cloud [9],

[2)] The values of masses (in Mev) and moments of inertia (in $10^{-3}\,\mathrm{Mev}^{-1}$) for the hedgehog with $B = 1$ and dipole configuration with $B = 2$ [9] in flavor symmetric (FS) and flavor symmetry broken (FSB) cases. $F_\pi = 186\,\mathrm{Mev}$, $e = 4.12$.

and the mass splittings are within $\sim 200 - 300$ Mev. More detailed calculations will be presented elsewhere.

4. To conclude, the quantization scheme for the $SU(3)$ skyrmions is presented and the quantization condition found previously [7] is generalized for skyrmions with arbitrary strangeness content. The relation (7) is valid for all known $B = 2$ local minima in $SU(3)$ configuration space shown in Fig.1. The moments of inertia of arbitrary $SU(3)$ skyrmions can be calculated with the help of formulas (10), (11).

For the dipole-type configuration with $SC = 0.5$ our results are in qualitative agreement with those obtained in [17] for interaction potential of two strange baryons located at large distances. The new branch of strange dibaryons additional to known previously [4], [11] is predicted with smallest uncertainty in the absolute values of masses due to the Casimir energy, relative to the corresponding $B = 1$ states. The prediction by chiral soliton models of the rich spectrum of baryonic states with different values of strangeness remains one of the intriguing properties of such models. It is difficult to observe these states, especially those which are above the threshold for the decay due to strong interaction. However, further investigations of the predictions of effective field theories providing new approach of the description of fundamental properties of matter are of interest.

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