RADIATIVE TAIL IN π_{e2} DECAY AND SOME COMMENTS TO μ–e UNIVERSALITY

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The result of lowest-order perturbation theory calculations of the photon and positron spectra in radiative π_{e2} decay are generalized to all orders of perturbation theory using the structure-function method. An additional source of radiative corrections to the ratio of the positron and muon channels of pion decay, due to emission of virtual and real photons and pairs, is considered. It depends on details of the detection of the final particles and is large enough to be taken into account in theoretical estimates with a level of accuracy of 0.1%.

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As a first step in the calculation of the spectra of radiative pion decays we reproduce the results obtained by Berman and Kinoshita [1], treating the pion as a point-like particle. Kinoshita [1] calculated the positron energy spectrum in radiative pion decay:

$$\frac{d\Gamma}{\Gamma_0 dy} = \frac{\alpha}{\pi}\left[\frac{1+y^2}{1-y}(L-1) - 1 + y - \frac{1}{2}(1-y)\ln(1-y) + \frac{1+y^2}{1-y}\ln y\right], \quad (1)$$

$$y_{\min} \leq y \leq 1 + m_e^2/m_\pi^2,$$

where $y = 2\varepsilon_e/m_\pi$ is the positron energy fraction, ε_e is the positron energy (here and below we have in mind the rest frame of the pion), $L = \ln(m_\pi/m_e) = 5.6$ is the "large logarithm", and m_π, m_e are the masses of the pion and positron. The quantity

$$\Gamma_0 = \frac{G^2|V_{ud}|^2}{8\pi}f_\pi^2 m_e^2 m_\pi\left(1 - \frac{m_e^2}{m_\pi^2}\right)^2 = 2.53\cdot10^{-14} \text{ MeV}, \quad (2)$$

is the total width of π_{e2} decay, calculated in the Born approximation.

We will now calculate the photon spectrum. Consider first the emission of a soft real photon. The corresponding contribution to the total width may be obtained by the standard integration of the differential widths:

$$\frac{d\Gamma^{\text{soft}}}{\Gamma_0} = -\frac{\alpha}{4\pi^2} \int \frac{d^3k}{\omega} \left(\frac{P}{Pk} - \frac{p_e}{p_e k} \right)^2 \Bigg|_{\omega \leq \Delta\varepsilon \ll m_\pi/2}, \tag{3}$$

where P, p_e, k are the four-momenta of the pion, positron, and photon, respectively, $P^2 = m_\pi^2$, $p_e^2 = m_e^2$, $k^2 = \lambda^2$, and λ is the photon mass. The result has the form

$$\begin{aligned}
\frac{\Gamma^{\text{soft}}}{\Gamma_0} &= \frac{\alpha}{\pi} \left[-b(\sigma) \ln \frac{2\Delta\varepsilon}{\lambda} + 1 - \frac{1+\sigma}{2(1-\sigma)} \ln \sigma - \right. \\
&\quad \left. - \frac{1+\sigma}{4(1-\sigma)} \ln^2 \sigma - \frac{1+\sigma}{1-\sigma} \text{Li}_2(1-\sigma) \right],
\end{aligned} \tag{4}$$

where

$$b(\sigma) = \frac{1+\sigma}{1-\sigma} \ln \sigma + 2, \quad \sigma = \frac{m_e^2}{m_\pi^2}, \quad \text{Li}_2(x) = -\int_0^x \frac{dt}{t} \ln(1-t). \tag{5}$$

Consider now the hard photon emission process

$$\pi^+(P) \to e^+(p_e) + \nu_e(p_\nu) + \gamma(k). \tag{6}$$

The standard procedure of final-states summing of the squared modulus of its matrix element and integration over the neutrino phase volume leads to the spectral distribution over the photon energy fraction $x = 2k^0/m_\pi$:

$$\begin{aligned}
\frac{d\Gamma}{\Gamma_0 dx} &= \frac{\alpha}{2\pi} \frac{x(1-x-\sigma)}{(1-\sigma)^2} \left[-\frac{4(1-\sigma)}{x^2} - \frac{1}{1-x} + \right. \\
&\quad \left. + \frac{1}{x(1-x-\sigma)} \left(\frac{1}{x}(1+(1-x)^2) + 2\sigma - 2\frac{\sigma^2}{x} \right) \ln \frac{1-x}{\sigma} \right].
\end{aligned} \tag{7}$$

Further integration of this spectrum give the result

$$\begin{aligned}
\int_{x_{\min}}^{1-\sigma} \frac{d\Gamma}{\Gamma_0 dx} dx &= \frac{\alpha}{2\pi} \left[-2b(\sigma) \ln \frac{1-\sigma}{x_{\min}} - 2\frac{1+\sigma}{1-\sigma} \text{Li}_2(1-\sigma) + \right. \\
&\quad \left. + \frac{3(1-2\sigma)}{2(1-\sigma)^2} \ln \sigma + \frac{19-25\sigma}{4(1-\sigma)} \right], \quad x_{\min} = \frac{2k_{\min}^0}{m_\pi}.
\end{aligned} \tag{8}$$

Putting $k_{\min}^0 = \Delta\varepsilon$ in this formula and adding the soft photon contribution, we obtain (in agreement with Kinoshita's 1959 result) the contribution to the width from the inner bremsstrahlung of a point-like pion:

$$\begin{aligned}
\frac{\Gamma_{IB}}{\Gamma_0} &= \frac{\alpha}{\pi} \left\{ b(\sigma) \left[\ln \frac{\lambda}{m_\pi} - \ln(1-\sigma) - \frac{1}{4} \ln \sigma + \frac{3}{4} \right] - \right. \\
&\quad \left. - 2\frac{1+\sigma}{1-\sigma} \text{Li}_2(1-\sigma) - \frac{\sigma(10-7\sigma)}{4(1-\sigma)^2} \ln \sigma + \frac{15-21\sigma}{8(1-\sigma)} \right\}.
\end{aligned} \tag{9}$$

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Now return to the positron spectrum. The contributions to it containing the large logarithm L may be associated with the known kernel of the Altarelli–Parisi–Lipatov evolution equation (see[2]):

$$P^{(1)}(y) = \lim_{\Delta \to 0} \left[\frac{1+y^2}{1-y}\theta(1-y-\Delta) + (\frac{3}{2} + 2\ln\Delta)\delta(1-y) \right] = \left(\frac{1+y^2}{1-y} \right)_+ . \tag{10}$$

Using the factorization theorem, we may generalize this spectrum to include the leading logarithmic terms in all orders of perturbation theory. This may be done in terms of structure functions $D(y,\sigma)$ [2]. In the case of the photon spectrum the function $D(1-x,\sigma)$ appears. The function $D(y,\sigma)$ describes the probability of finding a positron with energy fraction y inside the initial positron. It may be present in the form of a sum of non-singlet and singlet contributions, $D = D^\gamma + D^{e^+e^-}$. Iteration of the evolution equations gives

$$D(y,\sigma) = \delta(1-y) + P^{(1)}(y)\gamma + \frac{1}{2}(P^{(2)}(y) + P^{e^+e^-}(y))\gamma^2 + \ldots, \tag{11}$$

where

$$\gamma = -3\ln(1 - \frac{\alpha}{3\pi}(L-1)),$$

$$P^{(2)}(y) = \int_y^1 \frac{dt}{t} P^{(1)}(t) P^{(1)}(\frac{y}{t}) = \lim_{\Delta \to 0}\left\{ \left[(2\ln\Delta + \frac{3}{2})^2 - \right. \right.$$

$$- \frac{2\pi^2}{3} \right]\delta(1-y) + 2\left[\frac{1+y^2}{1-y}(2\ln(1-y) - \ln y + \frac{3}{2}) + \right.$$

$$\left. + \frac{1}{2}(1+y)\ln y - 1 + y \right]\theta(1-y-\Delta) \bigg\}, \tag{12}$$

$$P^{e^+e^-}(y) = \frac{2}{3}P^{(1)}(y) + \frac{(1-y)}{3y}(4 + 7y + 4y^2) + 2(1+y)\ln y.$$

It is convenient to use the smoothed form of them:

$$D^\gamma(y) = \frac{1}{2}b(1-y)^{\frac{1}{2}b-1}\left[1 + \frac{3}{2}b - \frac{1}{48}b^2(\frac{2}{3}L + \pi^2 - \frac{47}{8}) \right] - \tag{13}$$

$$- \frac{1}{4}b(1+y) + \frac{1}{32}b^2[4(1+y)\ln\frac{1}{1-y} + \frac{1+3y^2}{1-y}\ln\frac{1}{y} - 5 - y] + \mathcal{O}(b^3),$$

$$D^{e^+e^-}(y) = \frac{1}{3}\left(\frac{\alpha}{\pi}(L - \ln(1-y) - \frac{5}{6}) \right)^2 (1-y)^{\frac{1}{2}b-1} (1 + y^2 + \frac{1}{3}b(L - \ln(1-y) -$$

$$- \frac{5}{6})) + \frac{1}{96}b^2\left[\frac{1-y}{y}(4 + 7y + 4y^2) + 6(1+y)\ln y \right] + \mathcal{O}(b^3),$$

$$b = \frac{4\alpha}{\pi}(L-1).$$

The expressions for spectra are as follows:

$$\frac{d\Gamma}{\Gamma_0 dy} = D(y,\sigma)[1 + \frac{\alpha}{\pi}K_e(y)], \tag{14}$$

$$K_e(y) = -1 + y - \frac{1-y}{2}\ln(1-y) + \frac{1+y^2}{1-y}\ln y, \quad y = \frac{2\varepsilon_e}{m_\pi},$$

$$\frac{\mathrm{d}\Gamma}{\Gamma_0 \mathrm{d}x} = D(1-x,\sigma)[1+\frac{\alpha}{\pi}K_\gamma(x)],$$

$$K_\gamma(x) = x + \frac{1+(1-x)^2}{x}\ln(1-x), \quad x = \frac{2\omega}{m_\pi}.$$

Let us discuss the contribution of the inelastic processes considered above to the ratio of the widths of the positron and muon modes of pion decay, $R_{\pi_{12}}$:

$$R_{\tau l2} = \frac{\Gamma(\pi \to e\nu) + \Gamma(\pi \to e\nu\gamma)}{\Gamma(\pi \to \mu\nu) + \Gamma(\pi \to \mu\nu\gamma)}. \tag{15}$$

Close attention was paid to this quantity some years ago [3,4], but the corrections for emission processes in higher orders of perturbation theory were not taken into account. Keeping in mind that the quantity $P^{(1)}(y)$ has the property

$$\int_0^1 \mathrm{d}y \, P^{(1)}(y) = 0, \tag{16}$$

we make the important observation that as long as an experiment is proceeding in such a way that no cuts are imposed on the positron energy, then no large logarithmic contributions appear. However, if the cuts are such that the y integration is restricted or convoluted with a y-dependent function, some terms proportional to the large logarithm L will remain. We now suggest that there exists some minimum energy $\varepsilon_{\mathrm{th}}$ for detection of the positron. An additional contribution (not considered in [4]) appears:

$$\frac{\Delta R}{R_0} = -\frac{\alpha}{\pi} \int_0^{x_{\mathrm{th}}} \mathrm{d}x \, \frac{1+x^2}{1-x}(L-1) + \tag{17}$$

$$+ \frac{\alpha}{\pi} \int_{x_{\mathrm{th}}}^1 \mathrm{d}x \, D(x,\sigma) K_e(x), \quad x_{\mathrm{th}} = \frac{2\varepsilon_{\mathrm{th}}}{m_\pi},$$

where

$$R_0 = \frac{m_e^2}{m_\mu^2} \frac{(1-m_e^2/m_\pi^2)^2}{(1-m_\mu^2/m_\pi^2)^2} = 1.28347 \cdot 10^{-4}. \tag{18}$$

For typical values $x_{\mathrm{th}} = 0.1$ this additional contribution will have a magnitude of order 10^{-3} and should be taken into account in calculations for accuracy at the 0.1% level.

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