Pis'ma v ZhETF, vol.65, iss.5, pp.374 - 379

ON "HODGE" TOPOLOGICAL STRINGS AT GENUS ZERO

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Submitted 9 January 1997
Resubmitted 13 February 1997

The "Hodge strings" construction of solutions to associativity equations based on t-part of $t-t^*$ equations is proposed.This construction formalizes and generalizes the "integration over the position of the marked point" procedure for computation of amplitudes in topological conformal theories coupled to topological gravity.

PACS: 11.25.-w

1. Topological strings and associativity equation. The "topological string theory" [1−6] studies genus q "generalized amplitudes" GA_q taking values in cohomologies of the Deligne-Mamford compactification $\bar{M}_{q,n}$ of the moduli space of complex structures of genus q Riemann surfaces with n marked points. The pairing between GA_q and the cycle $C \in \bar{M}_{q,n}$ is given by [1, 2, 5] the functional integral:

$$(GA_q, C)(V_1, \ldots, V_n) = \int_{C \in \bar{M}_{q,n}} \int \mathcal{D}\phi V_1(\phi(z_1)) \ldots V_n(\phi(z_n)) \exp(S_{TS}(\phi)), \qquad (1)$$

fields $V_i(\phi(z))$ are called "vertex operators" and ordinary "amplitudes" $A_q(V_1, \ldots, V_n)$ correspond to $C = \bar{M}_{q,n}$.

Deligne-Mamford compactification $\bar{M}_{0,n}$ is a union of $M_{0,n}$ (set of n noncoincident points on CP_1 moduli $SL(2,C)$ action) and compactification divisor *Comp*. The divisor *Comp* is a union of components $C(S)$, where S is a partition of n marked points into two groups consisting of $n_1(S)$ and $n_2(S)$ points, $n_i > 1$. A surface corresponding to a general point in $C(S)$ is a union of two spheres having one common point with $n_1(S)$ marked points on the first sphere and $n_2(S)$ on the second. The set of general points in $C(S)$ form a space $M_{0,n_1+1} \otimes M_{0,n_2+1}$.

It is expected that the functional integral for surfaces corresponding to points in $C(S)$ factorizes and [1]

$$(GA_0, C(S))(V_{i_1}, \ldots, V_{i_n}) = \eta^{jk} A_0(V_{i_1}, \ldots, V_{i_{n_1}}, V_j) A_0(V_{i_{l+1}}, \ldots, V_{i_{n_2}}, V_k) \qquad (2)$$

where η is a matrix of symmetric bilinear nondegenerate product on vertex operators.

Keel found that the homology ring H_* of $\bar{M}_{0,k}$ is generated by cycles $C(S)$.He described relations between these cycles in homologies leading (due to (2)) to constraints on GA_0.

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An elegant way of formulation of these constraints uses the generating function for "amplitudes". Introducing formal parameters T_i we define

$$F(T) = \sum_{k=3}^{\infty} \frac{1}{k!} A_0(T_{i_1} V_{i_1}, \ldots, T_{i_k} V_{i_k}) \tag{3}$$

Then

$$\frac{\partial^3 F(T)}{\partial T_i \partial T_j \partial T_k} \eta^{kl} \frac{\partial^3 F(T)}{\partial T_l \partial T_p \partial T_q} = \frac{\partial^3 F(T)}{\partial T_i \partial T_p \partial T_k} \eta^{kl} \frac{\partial^3 F(T)}{\partial T_l \partial T_j \partial T_q} \tag{4}$$

Using factorization property and Keel's decription of homologies of the moduli space one can reconstruct GA_0 from A_0 [5], see also [4].

2. Amplitudes in topological conformal theory coupled to topological gravity. The "Hodge string" construction generalizes the "integration over the position of the marked point " procedure [1 − 4] of computation of amplitudes in "conformal topological theory coupled to topological gravity".

The general covariant action S_m of the topological field theory is a sum of a "topological"(metric independent) Q-closed term S_{top} and a Q-exact term for a fermionic scalar symmetry Q:

$$S_m = S_{top}(\phi) + Q(R(\phi), g),$$

where g denotes the metric on the Riemann surface. The energy-momentum tensor T is Q-exact:

$$T = Q(\frac{\delta R}{\delta g}) = Q(G) \tag{5}$$

We call topological field theory conformal, if R is conformal invariant, i.e. G is traceless.

We introduce fermionic two-tensor fields ψ, such that functions of g,ψ are forms on the space of metrics and external differential on these forms: $Q_g = \psi \frac{\delta}{\delta g}$.

The action for topological theory coupled to topological gravity is

$$S_{TS} = S_m + \psi G = S_{top} + (Q + Q_g)(R).$$

The functional integral $Z(g, \psi)$ over the set of fields ϕ with the action S_{TS} is a closed form on the space of metrics. Since G is traceless, Z is a horizontal [4, 8] [2)] form with respect to the action of conformal transformations of metric and diffeomorphisms of the Riemann surface, thus it defines a closed form on the moduli space of conformal(=complex) structures on the genus q Riemann surface.

To construct generalized amplitudes we insert at marked points on Riemann surface fields(zero-observables="vertex operators") V_i such that

$$Q(V_i) = 0, G_{0,-}(V_i) = 0. \tag{6}$$

Here $G_{0,-}$ is the superpartner of the component of the energy-momentum tensor $T_{0,-}$ that corresponds to the rotation with the constant phase $z \to e^{i\theta} z$ of the local coordinate at the marked point. First condition in (6) is needed to construct

[2)] Differential form on the principal bundle is called horizontal if its contraction with the vertical (tangent to fiber) vector is zero. Closed horizontal forms on the total space correspond to closed forms on the base of the bundle.

a closed form on the space of metrics while the second provides horizontality of the corresponding form with respect to diffeomorphisms that leave marked points fixed but rotate local coordinate [10, 4, 11, 8].

3. Integration over positions of marked points. The integration over marked points procedure reduces all genus zero amplitudes to the three point amplitude:

$$F_{ijk} = A_0(V_i, V_j, V_k),$$

that could be computed from the topological matter theory.

In conformal topological theory we associate to a zero observable V_i a two-observable $V_i^{(2)} = G_{L,-1}G_{R,-1}V_i$. Thus, we deform a topological theory to a family of theories parametrized by t, with the action $S_m(t) = S_m + t_i V_i^{(2)}$, so zero-observables V form a tangent bundle to this space of theories [4].

If in the functional integral that computes the measure on $M_{0,n}$ we first integrate over the position of the marked point and only then take the functional integral, the n-point amplitude becomes the derivative in t of the $n-1$ point amplitude.

In the process of integration we should take the special care about the region where the moving point tends to hit a fixed point since the geometry there is not a naive one. The contribution from this region(contact terms [3, 9, 8, 6]) leads to a specific contact term connection on the bundle of zero-observables over the space of theories and thus on the tangent space to the space of theories.

Repeating this procedure again and again we can recover amplitudes from $F_{ijk}(t)$. The amplitudes should be symmetric and independent of the order of integration over positions of marked points.

In other terms, generating parameters T from (3) should become so-called special coordinates on the space of theories, the derivatives with respect to special coordinates should become covariantly constant sections of the contact term connection and symmetric tensor F_{ijk} (in special coordinate frame) should be a third derivative of $F(T)$. Moreover, $F(T)$ has to satisfy eq.(4) WDVV equations (4).

This implies that the contact term connection is quite a special one!

To gain better understanding of this connection we will study the space of states in 2d theory associated to the boundary of Riemann surface - to the circle. Moreover, we will restrict ourselfs to the subspace H of this states that are invariant under the constant rotations of the circle.

Fermionic symmetry Q of the theory and $G_{0,-}$ reduce to odd anticommuting operators Q and G_- on H.

Zero-observables V_i being inserted at the middle of the punctured disc generate states h_i that are Q and G_- closed:

$$Qh_i = G_- h_i = 0, \tag{7}$$

the zero observable 1 generate the distinguished state h_0. The operation of sewing two discs together corresponds to the bilinear pairing $<,>$. Integrals of zero observables along the boundary give operators $\Phi_i = \int_{S_1} V_i d\sigma$.

One can show that they have the following properties:

$$Q^2 = G_-^2 = QG_- + G_-Q = 0, [Q, \Phi_i] = 0, [\Phi_i, \Phi_j] = 0, \tag{8}$$

$$Q^T = \epsilon Q, G^T = -\epsilon G, \Phi^T = \Phi \tag{9}$$

Here transposition "T" is taken with respect to the pairing $<,>$, and operator ϵ commutes with Φ and anticommutes with Q and G_-.

In the deformed theory $Q(t) = Q + [G_-, t_i\Phi_i]$ at first order in t. To ensure it globally we will take for simplicity[3)]

$$[[G_-, \Phi_i], \Phi_j] = 0. \tag{10}$$

The contribution from the region near the place where the "moving"i-th point hits the marked j-th one gives the "cancelled propagator argument"(CPA) constraint on states h_j over the space of theories [3, 8, 9]:

$$\delta_i^{(CPA)} h_j = G_- \int_0^\infty d\tau G_{0,+} \exp(-\tau T_{0,+}) \Phi_i h_j, \tag{11}$$

so $\delta^{(CPA)} h$ is G_--exact. Here $T_{0,+}$ is the Hamiltonian acting on the space H, and $G_{0,+}$ is its superpartner: $T_{0,+} = Q(G_{0,+})$.

Covariantly constant sections[4)] of the (CPA) connection will be denoted as $h_i(t)$. This connection induces the connection on the space of zero-observables: covariantly constant sections of contact term connections $V_i(t) = u_i^j(t) V_j$ are such that being inserted in the middle of the disc in the t-deformed theory they produce covariantly constant sections $h_i(t)$:

$$h_i(t) = lim_{r \to 0} r^{T_{0,+}} \Phi_j h_0(t) u_i^j(t). \tag{12}$$

Let us denote as $C_i(t)$ the matrix of action of Φ_i in $Q(t)$-cohomologies. Then the relation (12) reads:

$$[h_i(t)]_{Q(t)} = u_i^j(t) C_j(t) [h_0(t)]_{Q(t)} \tag{13}$$

here and below $[h]_Q$ stands for a class of a Q-closed element h in Q-cohomologies.

From the functional integral we get:

$$F_{ijk}(t) = < h_i(t), \Phi_l h_k(t) > u_j^l(t) \tag{14}$$

While the string origin of the described procedure is quite natural its consistency is far from being obvious.

In the next section we will show how to construct solutions to the associativity equations(thus, all GA_0)) from the "Hodge" data $(H, Q, G_-, \Phi_i, <>)$ if we assume the Hodge property and the Primitive element property. In particular, this would justify the consistency of the "integration over the positions of the marked point" procedure if the "Hodge" data was obtained from some topological conformal theory.

Hodge property: There is a set of Q and G_- closed vectors h_i such that classes $[h_i]_Q$ and $[h_i]_{G_-}$ form basises in Q and G_- cohomologies.

Primitive element property: There is a class $[h_0]_Q$ in Q-cohomologies such that the matrix $D_{ia} = C_{i,a}^b h_{0,b}$ is square and non-degenerate. Here indexes a label some basis in Q-cohomologies, $C_{i,a}^b$ is a matrix representing the action of Φ_i in these cohomologies, and $h_{0,b}$ are components of the class $[h_0]_Q$.

4. "Hodge string" construction. The "Hodge" string construction gives the solution to the associativity equations starting from the following data: Z_2 -graded

3) In general case one has to go in for Kodaira-Spencer type arguments, see [6].

4) Flatness of CPA connection is nessesary for the consistency of the procedure.

vector space H, odd operators Q and G_- , even operators Φ_i, and a bilinear pairing $<,>$, satisfying properties (8,9,10),the Hodge property[5)] and the Primitive element property.

The construction goes in two steps. In the first step we construct a flat connection with the spectral parameter from the Hodge data.In "physical" terms it is a constraint on the space of states The Primitive element property is not used in the first step. In the second step with the help of Primitive element property we induce flat constraint on the tangent bundle to the deformation space from the constraint constructed in the step one (i.e. we induce connection on the space of zero-observables from the connection on the space of states, like in (13)). Then we will integrate covariantly constant vector fields of this constraint to special coordinates T on the deformation space and finally construct $F(T)$.

Step 1. From the Hodge data one canonically constructs the connection(first constructed by K.Saito [12] in a slightly different context)

$$\frac{\partial}{\partial t_i}\delta_{ab} + z^{-1}C_{i,ab}(t) \tag{15}$$

such that this connection is flat for all z and $C_{i,ab} = C_{i,ba}$. This connection is known as the t-part of $t - t^*$ equations [7].

Idea of the proof: The Hodge Property leads to the Hodge Property for $Q(t)$ and G_- for all t close enough to zero(with the preferred vectors $h_i(t)$, such that $\partial_t h_i(t)$ is G_- exact - they generalize covariantly constant sections of CPA connection (11)). Consider $Q(t,z) = Q(t) + zG_-$ cohomologies in $H \otimes C[z,z^{-1}]$. Classes of $[P_i(z,z^{-1}h_i(t)]_{Q(t,z)}$ (for P_i being t-independent polynomials) form a "Hodge" basis in $Q(t,z)$ cohomologies. Next, we construct the "Gauss-Manin" (in the sence of K.Saito) flat connection in $Q(t,z)$ cohomologies through its covariantly constant sections

$$[\exp(\frac{-t_i\Phi_i}{z})]_{Q(t,z)}.$$

The "Gauss-Manin" constraint written in the "Hodge" basis takes the form of (15). Since bilinear pairing descends to G_- cohomologies it is t independent in the "Hodge" basis and can be taken to be equal to δ_{ab}. This leads to the symmetry of matrix C_i.

Step 2. From Step 1 we conclude that there exist a symmetric matrix τ_{ab}, such that

$$C_{i,ab} = \frac{\partial}{\partial t_i}\tau_{ab}.$$

Let us define special coordinates T_a on the deformation space with the help of the Primitive element

$$T_a(t) = \tau_{ab}(t)h_{0,b} \tag{16}$$

Statement: There exists a function $F(T)$ defined by

$$\frac{\partial}{\partial T_a \partial T_b}F(T) = \tau_{ab}(t(T)) \tag{17}$$

such that it satisfies the associativity equations with $\eta^{ab} = \delta^{ab}$.

[5)]The Hodge property is satisfied, for example, if Q and G_- are two supersymmetries in $N = 2$ supersymmetric quantum mechanics with the discrete spectrum of the Hamiltonian.

Proof: Explicite check.

Then we define a new set of coordinates[6] T_i as linear combinations of T_a by:

$$T_a = C_{i,ab}(0)h_{0,b} \tag{18}$$

A function $F(T_a(T_i))$ is the desired function that solves associativity equations with η^{ij},such that its inverse is given by:

$$(\eta^{-1})_{ij} =< h_0, C_iC_jh_0 >= h_{0,a}(C_iC_j)_{ab}h_{0,b}$$

Below we present some explicit formulas. Define

$$C_{i,ab}(t) = \sum C_{ij_1\dots j_n,ab}\frac{t_{j_1}\dots t_{j_n}}{n!}, F(T_i) = \sum F_{j_1\dots j_n}\frac{T_{j_1}\dots T_{j_n}}{n!} \tag{19}$$

Then

$$\begin{aligned} &F_{ijk} =< h_0, C_iC_jC_kh_0 >, F_{ijkl} =< h_0, C_i[C_j, C_{kl}]h_0 > \\ &F_{ijklm} =< h_0, C_i[C_{jkl}, C_m]h_0 > + \\ &< h_0[C_{im}, C_j]C_{kl}h_0 > + < h_0[C_{im}, C_l]C_{jk}h_0 > + < h_0[C_{im}, C_k]C_{lj}h_0 > \end{aligned} \tag{20}$$

I would like to thank R.Dijkgraaf, A.Gerasimov, G.Moore, N.Nekrasov, I.Polyubin, A.Rosly, S.Shatashvili for helpfull discussions.

This work was supported by grant RFFI-96-02-18046.

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[6]The coordinates T_i integrate vector fields u_i introduced in (12), (13).