Supplemental material to the article

Spatial distribution of electric field in a quantum superlattice with an injecting contact: exact solution

In this Supplementary we present the details of calculations of the spatial field profiles $f(x)$, Eqs. (6MT)–(8MT)[1], and the voltage-current characteristics $u(j)$, Eqs. (9MT) and (10MT). Everywhere in our derivations presented we employ the scaled variables and dimensionless parameters introduced in the main text.

1. Equations for spatial distributions of the electric field. Here we derive Eqs. (6MT) and (7MT) and demonstrate that both these expressions become identical in the case $j = 1/2$.

Our starting point is the differential equation for the electric field (4MT) with the initial condition $f(x = 0) = f_e$. After separation of the variables and integration we get

$$\alpha x = \int\limits_{f(0)}^{f(x)} \frac{f\,df}{j(1+f^2)-f}. \tag{1}$$

Next, we make a substitution $f = z + \frac{1}{2j}$ and rewrite (1) as a sum of two integrals

$$\alpha x = \int\limits_{z(0)}^{z(x)} \frac{4jz\,dz}{4j^2z^2-(1-4j^2)} + \int\limits_{z(0)}^{z(x)} \frac{2\,dz}{4j^2z^2-(1-4j^2)}. \tag{2}$$

Depending on the value of the current density j, the expression $1-4j^2$ can be either positive or negative, and therefore both integrals in Eq. (2) should be taken in the different ways.

For $j < j^* = 1/2$ Eq. (2) reads

$$\alpha x = \int\limits_{z(0)}^{z(x)} \frac{4jz\,dz}{4j^2z^2-b^2} + \int\limits_{z(0)}^{z(x)} \frac{2\,dz}{4j^2z^2-b^2}, \tag{3}$$

where $b = \sqrt{1-4j^2} > 0$. After introducing an additional substitution $\psi = 2jz/b$ and then taking the integrals in (3) we get

$$\alpha x = \frac{1}{2j}\log\left|\psi^2-1\right|\Big|_{\psi(0)}^{\psi(x)} + \frac{1}{2jb}\log\left|\frac{\psi-1}{\psi+1}\right|\Big|_{\psi(0)}^{\psi(x)}, \tag{4}$$

which in terms of j can be rewritten as

$$\alpha x = \frac{1}{2j}\left(\frac{1}{\sqrt{1-4j^2}}\log\left|\frac{1+\sqrt{1-4j^2}-2jf}{1-\sqrt{1-4j^2}-2jf}\right| + \log\left|\frac{4j\left(j-f+jf^2\right)}{1-4j^2}\right|\right)\Bigg|_{f(0)}^{f(x)}. \tag{5}$$

Simplifying (5) and taking into account that $f(0) = f_e$ we obtain

$$x_1(f) = \frac{1}{2\alpha j}\left(\frac{1}{\sqrt{1-4j^2}}\log\left|\frac{(1+\sqrt{1-4j^2}-2jf)(1-\sqrt{1-4j^2}-2jf_e)}{(1-\sqrt{1-4j^2}-2jf)(1+\sqrt{1-4j^2}-2jf_e)}\right| + \log\left|\frac{j-f+jf^2}{j-f_e+jf_e^2}\right|\right). \tag{6}$$

[1]The equation numbers with the letters MT, like (6MT), indicate corresponding formulas from the Main Text of the paper.

Since the arguments of both logarithms in (6) are always positive (see Sect. 2), one can drop the signs of the absolute value inside the log functions in (6), and thus obtain Eq. (6MT).

Now consider Eq. (2) for the case $j > j^* = 1/2$

$$\nu x = \int\limits_{z(0)}^{z(x)} \frac{4jz\,dz}{4j^2z^2 + c^2} + \int\limits_{z(0)}^{z(x)} \frac{2\,dz}{4j^2z^2 + c^2}, \tag{7}$$

where $c = \sqrt{4j^2 - 1} > 0$. After repeating the calculations described above one can obtain

$$\alpha x = \frac{1}{2j}\left[\frac{2}{\sqrt{4j^2-1}}\arctan\left(\frac{2jf-1}{\sqrt{4j^2-1}}\right) + \log\left|\frac{4j(j-f+jf^2)}{1-4j^2}\right|\right]\Bigg|_{f_e}^{f(x)} \tag{8}$$

and, eventually,

$$x_2(f) = \frac{1}{\alpha j\sqrt{4j^2-1}}\left[\arctan\left(\frac{2jf-1}{\sqrt{4j^2-1}}\right) - \arctan\left(\frac{2jf_e-1}{\sqrt{4j^2-1}}\right)\right] + \frac{1}{2\nu j}\log\left|\frac{j-f+jf^2}{j-f_e+jf_e^2}\right|. \tag{9}$$

Dropping the modulus sign in (9) due to the reasons discussed in Sec. 2 yields Eq. (7MT).

One should expect that the solutions $x_1(f)$ and $x_2(f)$, which are obtained for $j < j^*$ and $j > j^*$, respectively, coincide with each other in the limit $j \to j^*$. Simple analysis of Eqs. (6MT) and (7MT) indeed shows that

$$\lim_{j\to j^*-0} x_1(f) = \lim_{j\to j^*+0} x_2(f) = x_{j^*}(f),$$

where

$$x_{j^*}(f) = \frac{1}{\alpha}\left[\log\left(\frac{f-1}{f_e-1}\right)^2 + \frac{2(f-f_e)}{(f-1)(f_e-1)}\right]. \tag{10}$$

2. Analysis of the moduli problem. In this Section we demonstrate that the arguments of the logarithms in Eqs. (6) and (9) are always positive and also derive a compact form of the expression for $x_1(f)$, Eq. (8MT).

We start with consideration of Eq. (6), which for the case $j < j^*$ can be rewritten as

$$x_1(f) = \frac{1}{2\alpha j}\left(\frac{1}{\sqrt{1-4j^2}}\log\left|\frac{(f_+-f)(f_--f_e)}{(f_--f)(f_+-f_e)}\right| + \log\left|\frac{(f_+-f)(f_--f)}{(f_+-f_e)(f_--f_e)}\right|\right). \tag{11}$$

According to the classification given in the main text of the paper there are three different types of the spatial field profiles $f(x)$. Therefore our analysis of the arguments of log functions in Eq. (11) should be performed for each of these types separately:

- the first type of $f(x)$-profiles is determined by the boundary condition satisfying $f_e < f_-$. Since the stationary point f_- is an attractor, the electric field strength f remains less than f_-. Taking into account that $f_- < f_+$, one can see that the arguments of both logarithms are positive;
- for the second type of $f(x)$-profiles the boundary condition is $f_- < f_e < f_+$. The stationary point f_+ is unstable, and therefore the electric field strength is limited by the values f_+ and f_- so that $f_- < f < f_+$. The later conditions the positive value of the arguments of the log functions in (6);
- the $f(x)$ distributions, corresponding to the third type, always lie above the line $f = f_+$ (see Fig. 1d). In this case $f, f_e > f_+$, and again the arguments of the logarithms are positive.

Since Eq. (11) is equivalent to Eq. (6), we can also conclude that the arguments of the logarithms involved in (6) are positive as well. Note that after some simple transformation Eq. (11) can be simplified to the symmetric form (8MT).

Now consider the argument of the logarithm in Eq. (9)

$$r = \frac{j - f + jf^2}{j - f_e + jf_e^2}$$

for the case $j > j^* = 1/2$. Both numerator and denominator of r are quadratic functions of f and f_e, respectively. These quadratic equations have the same discriminant which value is negative for $j > 1/2$. Therefore r itself is positive.

3. Equations for the voltage drop across superlattice. In this Section we derive and analyze Eqs. (9MT), (10MT) for the dependence of the voltage dropped along the superlattices upon the electric current flowing through it. By integrating the basic equation $u = \int_0^1 f(x)dx$ by parts we obtain

$$u = f_c - \int_{f_e}^{f_c} x(f)\, df, \tag{12}$$

where f_c is defined by $x(f_c) = 1$. Substituting the expressions (6MT) and (7MT) into (12) and taking the corresponding integrals give us

$$u_1(j) = \frac{2(f_c - f_e) + \alpha}{2\alpha j} - \frac{1}{4\alpha j^2}\left[2\sqrt{1-4j^2}\,\mathrm{arctanh}\left(\frac{(f_c - f_e)\sqrt{1-4j^2}}{f_c + f_e - 2j(1 + f_c f_e)}\right) + \log\left(\frac{j - f_e + jf_e^2}{j - f_c + jf_c^2}\right)\right],$$

$$u_2(j) = \frac{f_c - f_e + \alpha}{\alpha j} - \frac{2}{\alpha\sqrt{4j^2-1}}\left[\arctan\left(\frac{2jf_c - 1}{\sqrt{4j^2-1}}\right) - \arctan\left(\frac{2jf_e - 1}{\sqrt{4j^2-1}}\right)\right], \tag{13}$$

where the inverse hyperbolic tangent function is $\mathrm{arctanh}(y) = \frac{1}{2}\log\left(\frac{1+y}{1-y}\right)$ $(y^2 < 1)$.

The above equations for $u_1(j)$ and $u_2(j)$, which are derived for the cases $j < j^*$ and $j > j^*$, respectively, must coincide with each other at $j < j^* = 1/2$. In order to check this, we consider the limits

$$\lim_{j \to j^* - 0} u_1 = 1 + \frac{2(f_c - f_e)}{\alpha} + \frac{1}{\alpha}\log\left(\frac{f_c - 1}{f_e - 1}\right)^2, \tag{14}$$

$$\lim_{j \to j^* + 0} u_2 = 2 + \frac{2(f_c - f_e)}{\alpha} - \frac{2(f_c - f_e)}{\alpha(f_c - 1)(f_e - 1)}. \tag{15}$$

Next, after substituting Eq. (10) into the condition $x_{j^*}(f_c) = 1$ we obtain the equality

$$\frac{1}{\alpha}\log\left(\frac{f_c - 1}{f_e - 1}\right)^2 + \frac{1}{\alpha}\frac{2(f_c - f_e)}{(f_c - 1)(f_e - 1)} = 1.$$

By using this equality, it is easy to see that the expressions (14) and (15) are indeed equivalent.

As a concluding remark, we would like to notice that our major analytical findings for electric field and voltage, Eqs. (6MT)–(9MT), were additionally verified by comparison with the results of numerical solution of the basic differential equation (4MT).