Supplemental Material to the article "Electron spectrum topology and giant van Hove singularities in cubic lattices"

Formal definition of density of state reads

$$\rho(\epsilon;\tau) = \frac{1}{N}\sum_{\mathbf{k}} \delta(\epsilon - t(\mathbf{k},\tau)), \tag{1}$$

where $t(\mathbf{k},\tau)$ is the spectrum within tight-binding approximation. We choose the signs of transfer integrals in the spectrum as follows

$$t_{\mathrm{sc}}(\mathbf{k},\tau) = -2(\cos k_x + \cos k_y + \cos k_z) + 4\tau(\cos k_y \cos k_z + \cos k_z \cos k_x + \cos k_x \cos k_y),$$

$$t_{\mathrm{bcc}}(\mathbf{k},\tau) = -8\cos k_x \cos k_y \cos k_z + 2\tau(\cos 2k_x + \cos 2k_y + \cos 2k_z),$$

$$t_{\mathrm{fcc}}(\mathbf{k},\tau) = -4(\cos k_y \cos k_z + \cos k_x \cos k_z + \cos k_x \cos k_y) + 2\tau(\cos 2k_x + \cos 2k_y + \cos 2k_z),$$

where lattice parameters are taken as unity. For the van Hove point $\bar{\mathbf{k}}$ we expand the spectrum

$$t(\mathbf{k}) = t(\bar{\mathbf{k}}) + \frac{1}{2}\sum_{ij} \frac{\partial^2 t(\bar{\mathbf{k}})}{\partial k_i \partial k_j}(k_i - \bar{k}_i)(k_j - \bar{k}_j). \tag{2}$$

Let a_i be the eigenvalues of the matrix $\partial^2 t(\bar{\mathbf{k}})/\partial k_i \partial k_j$ (inverse mass tensor). We introduce the signature as the difference between numbers of positive and negative eigenvalues of mass tensor and write it in slashes, e.g., minimum (maximum) corresponds to $/+3/(/-3/)$, saddle points to $/\pm 1/$. For three-dimensional lattice in the non-degenerate case a local minimum (maximum) of the spectrum $t(\mathbf{k})$ corresponds to one-side square-root increase (decrease) of DOS as the energy ϵ deviates from ϵ_0:

$$\rho(\epsilon) = \rho(\epsilon_0) + A\sqrt{\theta(\pm(\epsilon-\epsilon_0))|\epsilon-\epsilon_0|} + O(\epsilon-\epsilon_0) \tag{3}$$

in the vicinity of the van Hove level ϵ_0; a saddle-type van Hove point with the mass tensor signature $/+1/$ $(/-1/)$ corresponds to one-side square-root decreasing,

$$\rho(\epsilon) = \rho(\epsilon_0) - A\sqrt{\theta(\mp(\epsilon-\epsilon_0))|\epsilon-\epsilon_0|} + O(\epsilon-\epsilon_0) \tag{4}$$

with positive constant $A = 2\pi V_{\mathrm{BZ}}|a_1 a_2 a_3|^{-3/2}$, V_{BZ} being the Brillouin zone volume.

Figure S1 shows the spectrum for SC and BCC lattice in high-symmetry directions of the Brillouin zone for τ being in the vicinity of topological transition.

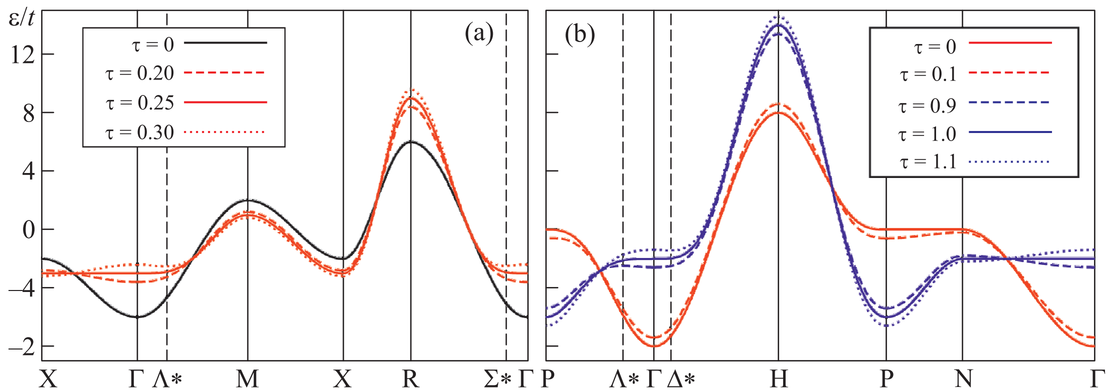

Fig. S1. Energy spectrum in high-symmetry directions of the Brillouin zone ("spaghetti"). SC (a), BCC (b) lattice, van Hove $\mathbf{k}$ points are shown

Table 1. Non-equivalent $\mathbf{k}$ points of van Hove singularities for SC lattice, see Fig. 1a of main text, $\tau \geq 0$. Arrow denotes a change of signature of mass tensor of van Hove $\mathbf{k}$ point as τ increases above these value $\tau_*^{\rm sc} = 1/4$. $k_{\Sigma^*}^{\rm sc} = \arccos\left[(2\tau)^{-1} - 1\right], k_{\Lambda^*}^{\rm sc} = \arccos\left[(4\tau)^{-1}\right]$

$\mathbf{k}$	$w^{\rm sc} = t(\mathbf{k})$	Inverse masses	Signature
$\Gamma(0,0,0)$	$-6+12\tau$	$2(1-4\tau), 2(1-4\tau), 2(1-4\tau)$	$\min \overset{\tau=1/4}{\rightarrow} \max$
$\mathrm{R}(\pi,\pi,\pi)$	$+6+12\tau$	$-2(1+4\tau), -2(1+4\tau), -2(1+4\tau)$	
$\mathrm{X}(0,0,\pi)$	$-2-4\tau$	$2, 2, 2(4\tau-1)$	$/+1/ \overset{\tau=1/4}{\rightarrow} \min$
$\mathrm{M}(0,\pi,\pi)$	$+2-4\tau$	$-2, -2, 2(4\tau+1);$	$/-1/$
$\Sigma^*(0, k_{\Sigma^*}^{\rm sc}, k_{\Sigma^*}^{\rm sc})$	$-\tau^{-1}+2-4\tau$	$2(4\tau-1), 4-\tau^{-1}, \tau^{-1}-4$	$\tau > 1/4, /+1/$
$\Lambda^*(k_{\Lambda^*}^{\rm sc}, k_{\Lambda^*}^{\rm sc}, k_{\Lambda^*}^{\rm sc})$	$-3\tau^{-1}/4$	$\tau^{-1}((4\tau)^2-1), (2\tau)^{-1}(1-(4\tau)^2), (2\tau)^{-1}(1-(4\tau)^2)$	$\tau > 1/4,\ /-1/$

Table 2. Non-equivalent $\mathbf{k}$-points of van Hove singularities for BCC lattice, see Fig. 1b of main text, $\tau \geq 0$. The notations are the same as in the Table 1. $k_{\Lambda^*}^{\rm bcc} = \arccos\tau, k_{\Delta^*}^{\rm bcc} = \arccos\tau^{-1}$

$\mathbf{k}$	$w^{\rm bcc} = t(\mathbf{k})$	Inverse masses	Signature
$\Gamma(0,0,0)$	$-8+6\tau$	$8(1-\tau), 8(1-\tau), 8(1-\tau)$	$\min \overset{\tau=1}{\rightarrow} \max$
$\mathrm{H}(0,0,\pi)$	$+8+6\tau$	$-8(1+\tau), -8(1+\tau), -8(1+\tau)$	max
$\mathrm{P}(\pi/2,\pi/2,\pi/2)$	-6τ	$8\tau, 8\tau, 8\tau$	min
$\mathrm{N}(\pi/2,\pi/2,0)$	-2τ	$-4\tau, 4(1+\tau), 4(-1+\tau)$	$/-1/ \overset{\tau=1}{\rightarrow} /+1/$
$\Lambda^*\left(k_{\Lambda^*}^{\rm bcc}, k_{\Lambda^*}^{\rm bcc}, k_{\Lambda^*}^{\rm bcc}\right)$	$2\tau(2\tau^2-3)$	$16\tau(1-\tau^2), 16\tau(1-\tau^2), -8\tau(1-\tau^2)$	$\tau < 1,\ /+1/$
$\Delta^*\left(0,0,k_{\Delta^*}^{\rm bcc}\right)$	$2\tau - 4\tau^{-1}$	$8(\tau^{-1}-\tau), 8(\tau^{-1}-\tau), 8(\tau-\tau^{-1})$	$\tau > 1,\ /-1/$

Consider exact expressions for the density of states in the SC and BCC lattice within the tight-binding approximation (see main text). The DOS can be presented as a sum of three contributions

$$\rho(\epsilon;\tau) = \mathcal{R}_\psi(\epsilon,\tau) + \mathcal{R}_{\varphi'}(\epsilon,\tau) + \mathcal{R}_\varphi(\epsilon,\tau). \tag{5}$$

For SC lattice, depending on τ value the contributions $\mathcal{R}_i^{\rm sc}$ read (arguments are omitted for brevity)

1. $\tau \leq 1/4$. $\mathcal{R}_\psi^{\rm sc} = \mathcal{R}_{\varphi'}^{\rm sc} = 0$

$$\mathcal{R}_\varphi^{\rm sc} = \begin{cases} \Phi_{\rm sc}(x_{\psi 1}^{\rm sc}, +1), & w_\Gamma^{\rm sc} < \epsilon < w_{\rm X}^{\rm sc}(\tau) \\ \Phi_{\rm sc}(-1, +1), & w_{\rm X}^{\rm sc}(\tau) < \epsilon < w_{\rm M}^{\rm sc}(\tau) \\ \Phi_{\rm sc}(-1, x_{\psi 2}^{\rm sc}), & w_{\rm M}^{\rm sc}(\tau) < \epsilon < w_{\rm R}^{\rm sc}. \end{cases} \tag{6}$$

2. $1/4 < \tau \leq 1/2$.

$$\mathcal{R}_\psi^{\rm sc} = \begin{cases} 2\Psi_{\rm sc}(x_\varphi^{\rm sc}, +1), & w_{\rm X}^{\rm sc} < \epsilon < w_{\Sigma^*}^{\rm sc} \\ 2\Psi_{\rm sc}(x_{\zeta-}^{\rm sc}, x_{\zeta+}^{\rm sc}), & w_{\Sigma^*}^{\rm sc} < \epsilon < w_{\Lambda^*}^{\rm sc}, \end{cases} \tag{7}$$

$$\mathcal{R}_{\varphi'}^{\rm sc} = \begin{cases} 2[\Phi_{\rm sc}(x_{\psi 1}^{\rm sc}, x_{\zeta-}^{\rm sc}) + \Phi_{\rm sc}(x_{\zeta+}^{\rm sc}, +1)], & w_{\Sigma^*}^{\rm sc} < \epsilon < w_{\Lambda^*}^{\rm sc} \\ 2\Phi_{\rm sc}(x_{\psi 1}^{\rm sc}, +1), & w_{\Lambda^*}^{\rm sc} < \epsilon < w_\Gamma^{\rm sc}, \end{cases} \tag{8}$$

$$\mathcal{R}_\varphi^{\rm sc} = \begin{cases} \Phi_{\rm sc}(-1, x_{\psi 1}^{\rm sc}), & w_{\rm X}^{\rm sc} < \epsilon < w_\Gamma^{\rm sc} \\ \Phi_{\rm sc}(-1, +1), & w_\Gamma^{\rm sc} < \epsilon < w_{\rm M}^{\rm sc} \\ \Phi_{\rm sc}(-1, x_{\psi 2}^{\rm sc}), & w_{\rm M}^{\rm sc} < \epsilon < w_{\rm R}^{\rm sc}. \end{cases} \tag{9}$$

Table 3. Non-equivalent $\mathbf{k}$ points of van Hove singularities for FCC lattice, see Fig. 2a of main text. The notations are the same as in the Table 1. $k_{\Sigma^*}^{\rm fcc} = \arccos(2\tau-1)^{-1}, k_{\Delta^*}^{\rm fcc} = \arccos\tau^{-1}$. $a_{\Delta^*1}^{\rm fcc} = 4(\tau^{-1}-1)(1+2\tau), a_{\Delta^*2}^{\rm fcc} = 8\tau^{-1}(\tau^2-1), a_{\Sigma^*1}^{\rm fcc} = 8(\tau-1)(1+2\tau)/(1-2\tau), a_{\Sigma^*2}^{\rm fcc} = 16\tau(\tau-1)(1+2\tau)(1-2\tau)^{-2}, a_{\Sigma^*3}^{\rm fcc} = 16\tau(\tau-1)(2\tau-1)^{-1}$

$\mathbf{k}$	$w^{\rm fcc} = t(\mathbf{k})$	Inverse masses	Signature
$\Gamma(0,0,0)$	$-12+6\tau$	$8(1-\tau), 8(1-\tau), 8(1-\tau)$	$\min \overset{\tau=1}{\rightarrow} \max$
$\mathrm{X}(0,0,\pi)$	$+4+6\tau$	$-8\tau, -8\tau, -8(1+\tau)$	$\min \overset{\tau=-1}{\rightarrow} /+1/ \overset{\tau=0}{\rightarrow} \max$
$\mathrm{W}(0,\pi/2,\pi)$	$+4+2\tau$	$-4(1+2\tau), 8\tau, 8\tau$	$/+1/ \overset{\tau=-1/2}{\rightarrow} \max \overset{\tau=0}{\rightarrow} /-1/$
$\mathrm{L}(\pi/2,\pi/2,\pi/2)$	-6τ	$4(1+2\tau), 4(1+2\tau), 8(\tau-1)$	$\max \overset{\tau=-1/2}{\rightarrow} /+1/ \overset{\tau=+1}{\rightarrow} \min$
$\Delta^*\left(0,0,k_{\Delta^*}^{\rm fcc}\right)$	$-2\tau^{-1}(2+2\tau-\tau^2)$	$a_{\Delta^*1}^{\rm fcc}, a_{\Delta^*1}^{\rm fcc}, a_{\Delta^*2}^{\rm fcc}$	$/-1/, \tau > +1;\ /+1/, \tau < -1$
$\Sigma^*\left(0, k_{\Sigma^*}^{\rm fcc}, k_{\Sigma^*}^{\rm fcc}\right)$	$4(1-2\tau)^{-1} - 2\tau$	$a_{\Sigma^*1}^{\rm fcc}, a_{\Sigma^*2}^{\rm fcc}, a_{\Sigma^*3}^{\rm fcc}$	$/+1/, \tau > +1;\ /-1/, \tau < 0$

3. $\tau > 1/2$. The kinks of the functions $\mathcal{R}^{\rm sc}_{\varphi}$ and $\mathcal{R}^{\rm sc}_{\varphi'}$ at $\epsilon = w^{\rm sc}_0(\tau) = 4\tau - \tau^{-1}$ cancel each other.

$$\mathcal{R}^{\rm sc}_{\psi} = \begin{cases} 2\Psi_{\rm sc}\left(x^{\rm sc}_{\varphi}, +1\right), & w^{\rm sc}_{\rm X} < \epsilon < w^{\rm sc}_{\Sigma^*} \\ 2\Psi_{\rm sc}\left(x^{\rm sc}_{\zeta-}, x^{\rm sc}_{\zeta+}\right), & w^{\rm sc}_{\Sigma^*} < \epsilon < w^{\rm sc}_{\Lambda^*}, \end{cases} \tag{10}$$

$$\mathcal{R}^{\rm sc}_{\varphi'} = \begin{cases} 2\Phi_{\rm sc}\left(x^{\rm sc}_{\zeta+}, \min[x^{\rm sc}_{\psi 2}, +1]\right) + 2\Phi_{\rm sc}\left(x^{\rm sc}_{\psi 1}, x^{\rm sc}_{\zeta-}\right), & w^{\rm sc}_{\Sigma^*} < \epsilon < w^{\rm sc}_{\Lambda^*} \\ 2\Phi_{\rm sc}\left(x^{\rm sc}_{\psi 1}, \min[x^{\rm sc}_{\psi 2}, +1]\right), & w^{\rm sc}_{\Lambda^*} < \epsilon < w^{\rm sc}_{0}, \end{cases} \tag{11}$$

$$\mathcal{R}^{\rm sc}_{\varphi} = \begin{cases} \Phi_{\rm sc}\left(-1, x^{\rm sc}_{\psi 1}\right), & w^{\rm sc}_{\rm X} < \epsilon < w^{\rm sc}_{\rm M} \\ \Phi_{\rm sc}\left(-1, x^{\rm sc}_{\psi 1}\right) + \Phi_{\rm sc}\left(x^{\rm sc}_{\psi 2}, +1\right), & w^{\rm sc}_{\rm M} < \epsilon < w^{\rm sc}_{0} \\ \Phi_{\rm sc}\left(-1, x^{\rm sc}_{\psi 2}\right) + \Phi_{\rm sc}\left(x^{\rm sc}_{\psi 1}, +1\right), & w^{\rm sc}_{0} < \epsilon < w^{\rm sc}_{\Gamma} \\ \Phi_{\rm sc}\left(-1, x^{\rm sc}_{\psi 2}\right), & w^{\rm sc}_{\Gamma} < \epsilon < w^{\rm sc}_{\rm R}. \end{cases} \tag{12}$$

In these equations $F(x,y) = K(1-y/x)/\sqrt{x}$ is symmetric function, which is infinite when x or y tends to zero,

$$K(m) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - m\sin^2\phi}}$$

being full elliptic integral of the first kind

$$\Psi_{\rm sc}(x_1, x_2; \epsilon, \tau) = \frac{2}{\pi^3}\int\limits_{x_1}^{x_2} \frac{dx\, F(-\zeta_{\rm sc}(x;\epsilon,\tau), \psi_{\rm sc}(x;\epsilon,\tau))}{\sqrt{1-x^2}}, \tag{13}$$

$$\Phi_{\rm sc}(x_1, x_2; \epsilon, \tau) = \frac{2}{\pi^3}\int\limits_{x_1}^{x_2} \frac{dx\, F(\zeta_{\rm sc}(x;\epsilon,\tau), \varphi_{\rm sc}(x;\epsilon,\tau))}{\sqrt{1-x^2}}, \tag{14}$$

where

$$\zeta_{\rm sc}(x;\epsilon,\tau) = 16(\tau(\epsilon + 2x) + (1 - 2\tau x)^2), \tag{15}$$
$$\varphi_{\rm sc}(x;\epsilon,\tau) = (\epsilon + 2x + 4\tau)^2, \tag{16}$$
$$\psi_{\rm sc}(x;\epsilon,\tau) = (\epsilon + 2x - 4\tau)^2 - 16(1 - 2\tau x)^2, \tag{17}$$

the x integration bounds are $x^{\rm sc}_{\psi s}(\epsilon,\tau) = (1/2)(4((-1)^s + \tau) - \epsilon)/(1 + 4(-1)^s\tau)$, $s = 1, 2$; $x^{\rm sc}_{\varphi}(\epsilon,\tau) = -2\tau - \epsilon/2$, $x^{\rm sc}_{\zeta\pm}(\epsilon,\tau) = (\frac{1}{2} \pm \sqrt{-\frac{3}{4} - \tau\epsilon})/2\tau$, at $\epsilon < w^{\rm sc}_{\Lambda^*}(\tau)$. Plots of DOS for different values of τ for SC lattice are presented in Fig. S2a.

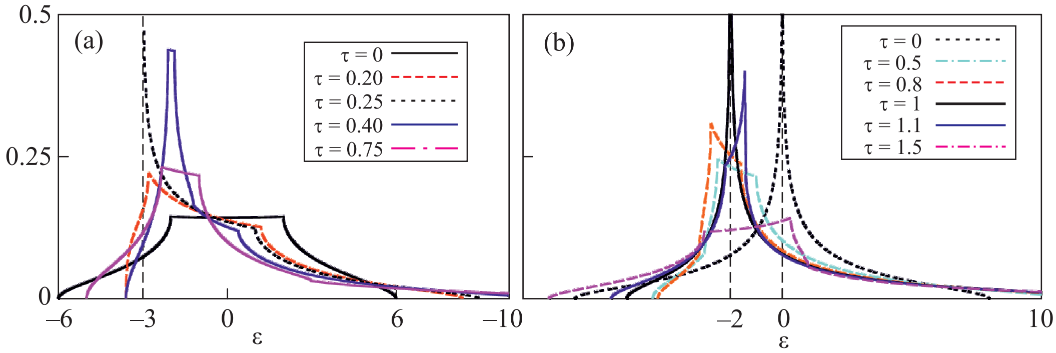

Fig. S2. DOS $t\rho(\epsilon)$ for SC (a) and BCC (b) lattices at different τ. Vertical dashed lines show the position of giant van Hove singularities

For BCC lattice we present the following form for the contributions $\mathcal{R}^{\rm bcc}_i$

1. $\tau \le 1$.

$$\mathcal{R}^{\rm bcc}_{\psi} = \begin{cases} 2\Psi_{\rm bcc}(0, x^{\rm bcc}_{\varphi}), & w^{\rm bcc}_{\rm P} < \epsilon < w^{\rm bcc}_{\Lambda^*} \\ 2\Psi_{\rm bcc}(0, x^{\rm bcc}_{\zeta}), & w^{\rm bcc}_{\Lambda^*} < \epsilon < w^{\rm bcc}_{\rm N}, \end{cases} \tag{18}$$

$$\mathcal{R}^{\rm bcc}_{\varphi'} = \begin{cases} 2\Phi_{\rm bcc}(x^{\rm bcc}_\zeta, x^{\rm bcc}_{\psi 1}), & w^{\rm bcc}_{\Lambda^*} < \epsilon < w^{\rm bcc}_{\rm N} \\ 2\Phi_{\rm bcc}(0, x^{\rm bcc}_{\psi 1}), & w^{\rm bcc}_{\rm N} < \epsilon < w^{\rm bcc}_0, \end{cases} \tag{19}$$

$$\mathcal{R}^{\rm bcc}_{\varphi} = \Phi_{\rm bcc}(x^{\rm bcc}_{\psi 1}, +1). \tag{20}$$

2. $\tau > 1$.

$$\mathcal{R}^{\rm bcc}_{\psi} = \begin{cases} 2\Psi_{\rm bcc}(0, x^{\rm bcc}_\varphi), & w^{\rm bcc}_{\rm P} < \epsilon < w^{\rm bcc}_{\rm N} \\ 2\Psi_{\rm bcc}(x^{\rm bcc}_\zeta, +1), & w^{\rm bcc}_{\rm N} < \epsilon < w^{\rm bcc}_{\Delta^*}, \end{cases} \tag{21}$$

$$\mathcal{R}^{\rm bcc}_{\varphi'} = \begin{cases} 2\Phi_{\rm bcc}(0, x^{\rm bcc}_\zeta), & w^{\rm bcc}_{\rm N} < \epsilon < w^{\rm bcc}_{\Delta^*} \\ 2\Phi_{\rm bcc}(0, x^{\rm bcc}_{\psi 1}) & \\ \quad +2\Phi_{\rm bcc}(x^{\rm bcc}_{\psi 2}, +1), & w^{\rm bcc}_{\Delta^*} < \epsilon < w^{\rm bcc}_{\Gamma}. \end{cases} \tag{22}$$

$$\mathcal{R}^{\rm bcc}_{\varphi} = \begin{cases} \Phi_{\rm bcc}(x^{\rm bcc}_{\psi 1}, x^{\rm bcc}_{\psi 2}), & w^{\rm bcc}_{\Delta^*} < \epsilon < w^{\rm bcc}_{\Gamma} \\ \Phi_{\rm bcc}(x^{\rm bcc}_{\psi 1}, +1), & w^{\rm bcc}_{\Gamma} < \epsilon < w^{\rm bcc}_{\rm H}, \end{cases} \tag{23}$$

The kinks of the functions $\mathcal{R}^{\rm bcc}_{\varphi}, \mathcal{R}^{\rm bcc}_{\varphi'}$ at $\epsilon = w^{\rm bcc}_0 = 2\tau$ cancel each other.

$$\Psi_{\rm bcc}(x_1, x_2; \epsilon, \tau) = \frac{2}{\pi^3} \int_{x_1}^{x_2} \frac{dx F(-\zeta_{\rm bcc}(x;\epsilon,\tau), \psi_{\rm bcc}(x;\epsilon,\tau))}{\sqrt{x(1-x)}}, \tag{24}$$

$$\Phi_{\rm bcc}(x_1, x_2; \epsilon, \tau) = \frac{2}{\pi^3} \int_{x_1}^{x_2} \frac{dx F(\zeta_{\rm bcc}(x;\epsilon,\tau), \varphi_{\rm bcc}(x;\epsilon,\tau))}{\sqrt{x(1-x)}}, \tag{25}$$

where

$$\zeta_{\rm bcc}(x;\epsilon,\tau) = 16(\tau(\epsilon+2\tau) + 4(1-\tau^2)x), \tag{26}$$

$$\varphi_{\rm bcc}(x;\epsilon,\tau) = (6\tau + \epsilon - 4\tau x)^2, \tag{27}$$

$$\psi_{\rm bcc}(x;\epsilon,\tau) = (\epsilon - 2\tau - 4\tau x)^2 - 64x, \tag{28}$$

and the x integration boundaries read $x^{\rm bcc}_\varphi(\epsilon,\tau) = (\epsilon+6\tau)/4\tau$, at $w^{\rm bcc}_{\rm P} < \epsilon$, $x^{\rm bcc}_\zeta(\epsilon,\tau) = \tau(\epsilon+2\tau)/4(\tau^2-1)$, at $(\epsilon - w^{\rm bcc}_\zeta)(\tau - 1) > 0$, $x^{\rm bcc}_{\psi 1,2}(\epsilon,\tau) = \left(\mp 2 + \sqrt{\tau(\epsilon-2\tau)+4}\right)^2/(4\tau^2)$, at $\epsilon > w^{\rm bcc}_{\Delta^*}$. DOS plots for different values of τ for BCC lattice are shown in Fig. S2b.

Full understanding for the dependence of DOS on τ can be obtained using the τ dependence of $\rho(w^{\rm vHS}_i(\tau),\tau)$ at the levels of van Hove points $\epsilon = w^{\rm vHS}_i(\tau)$. These plots are shown in Fig. S3. For SC lattice, the maximal value of DOS at $\tau > 1/4$ is achieved at the levels $\epsilon = w^{\rm sc}_{\Sigma^*}$ and $w^{\rm sc}_{\Lambda^*}$. These values slowly decrease as τ is shifted from 1/4 ($\rho_{\rm sc}(w^{\rm sc}_{\Sigma^*}(\tau),\tau) \approx \rho_{\rm sc}(w^{\rm sc}_{\Sigma^*}(\tau),\tau) \sim (\tau - 1/4)^{-1/2}$). For BCC lattice, the maximal value of DOS is always achieved at the energy level corresponding to inner points of the Brillouin zone: $\epsilon = w^{\rm bcc}_{\Lambda^*}$ at $\tau < 1$, $\epsilon = w^{\rm bcc}_{\Delta^*}$ at $\tau > 1$.

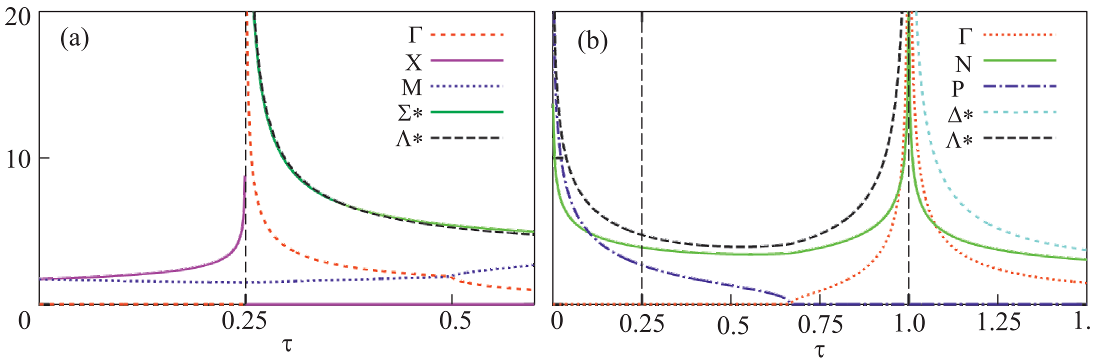

Fig. S3. τ dependence of DOS $W\rho(\epsilon;\tau)$ at each van Hove singularity level for SC (a) and BCC (b) lattices (W is the bandwidth). In the case (a), DOS values at $w^{\rm sc}_{\Lambda^*}$ and $w^{\rm sc}_{\Sigma^*}$ are almost equal, see main text