Supplementary Material to the article

"A Compatible system of equations related to the Lie superalgebra gl(n|m) and integrable Calogero–Moser model"

Compatibility of the systems. At first let us check that the defined Dynamical equations is a compatible system – so one have to check that the following commutator vanishes (for brevity denote $\partial_a = \partial_{\lambda_a}$)

$$\left[\kappa\partial_a - \sum_j z_j e_{aa}^{(j)} - \sum_{c,\neq a}(-1)^{p(c)}\frac{E_{ac}E_{ca}-E_{aa}}{\lambda_a-\lambda_c}, \kappa\partial_b - \sum_l z_l e_{bb}^{(l)} - \sum_{d,\neq b}(-1)^{p(d)}\frac{E_{bd}E_{db}-E_{bb}}{\lambda_b-\lambda_d}\right] = 0. \qquad \text{(S1)}$$

There are three types of summands

(1)

$$\left[\partial_a, \sum_{d,\neq b}(-1)^{p(d)}\frac{E_{bd}E_{db}-E_{bb}}{\lambda_b-\lambda_d}\right] + \left[\sum_{c,\neq a}(-1)^{p(c)}\frac{E_{ac}E_{ca}-E_{aa}}{\lambda_a-\lambda_c}, \partial_b\right] =$$
$$= (-1)^{p(a)}\frac{E_{ba}E_{ab}-E_{bb}}{(\lambda_b-\lambda_a)^2} - (-1)^{p(b)}\frac{E_{ab}E_{ba}-E_{aa}}{(\lambda_a-\lambda_b)^2} = 0.$$

The last equation is due to algebra relations

(2)

$$\left[\sum_j z_j e_{aa}^{(j)}, \sum_{d,\neq b}(-1)^{p(d)}\frac{E_{bd}E_{db}-E_{bb}}{\lambda_b-\lambda_d}\right] + \left[\sum_{c,\neq a}(-1)^{p(c)}\frac{E_{ac}E_{ca}-E_{aa}}{\lambda_a-\lambda_c}, \sum_l z_l e_{bb}^{(l)}\right] =$$
$$= \frac{(-1)^{p(a)}}{\lambda_{ba}}\left(\sum_j z_j(E_{ba}e_{ab}^{(j)} - e_{ba}^{(j)}E_{ab})\right) + \frac{(-1)^{p(b)}}{\lambda_{ab}}\left(\sum_j z_j(e_{ab}^{(j)}E_{ba} - E_{ab}e_{ba}^{(j)})\right) = 0.$$

The last is true due to the supersymmetric property of the tensor product

(3)

$$\left[\sum_{c,\neq a}(-1)^{p(c)}\frac{E_{ac}E_{ca}-E_{aa}}{\lambda_a-\lambda_c}, \sum_{d,\neq b}(-1)^{p(d)}\frac{E_{bd}E_{db}-E_{bb}}{\lambda_b-\lambda_d}\right] =$$
$$= \sum_{c,\neq a,b}\left(\left[(-1)^{p(b)}\frac{E_{ab}E_{ba}}{\lambda_{ab}}, (-1)^{p(c)}\frac{E_{bc}E_{cb}}{\lambda_{bc}}\right] + \left[\frac{E_{ac}E_{ca}}{\lambda_{ac}}, \frac{E_{bc}E_{cb}}{\lambda_{bc}}\right] + \right.$$
$$\left. + \left[(-1)^{p(c)}\frac{E_{ac}E_{ca}}{\lambda_{ac}}, (-1)^{p(a)}\frac{E_{ba}E_{ab}}{\lambda_{ba}}\right]\right) + (-1)^{p(a)+p(b)}\left[\frac{E_{ab}E_{ba}}{\lambda_{ab}}, \frac{E_{ba}E_{ab}}{\lambda_{ba}}\right].$$

A little calculation shows that this commutator is 0. Let us focus on the middle line of the above expression, which is the most difficult one. In order to prove the vanishing one have to consider several similar cases. As an example let us pick the following

$\mathbf{p(a) = p(b) = 0, \ p(c) = 1}$

$$-\left[\frac{E_{ab}E_{ba}}{\lambda_{ab}}, \frac{E_{bc}E_{cb}}{\lambda_{bc}}\right] + \left[\frac{E_{ac}E_{ca}}{\lambda_{ac}}, \frac{E_{bc}E_{cb}}{\lambda_{bc}}\right] - \left[\frac{E_{ac}E_{ca}}{\lambda_{ac}}, \frac{E_{ba}E_{ab}}{\lambda_{ba}}\right] =$$
$$= -\frac{E_{ac}E_{ba}E_{cb} - E_{bc}E_{ab}E_{ca}}{\lambda_{ab}\lambda_{bc}} + \frac{E_{ac}E_{ba}E_{cb} - E_{bc}E_{ab}E_{ca}}{\lambda_{ac}\lambda_{bc}} - \frac{E_{ac}E_{ba}E_{cb} - E_{bc}E_{ab}E_{ca}}{\lambda_{ac}\lambda_{ba}} = 0.$$

$\mathbf{p(b) = p(c) = 1,\ p(a) = 0}$

$$\left[\frac{E_{ab}E_{ba}}{\lambda_{ab}}, \frac{E_{bc}E_{cb}}{\lambda_{bc}}\right] + \left[\frac{E_{ac}E_{ca}}{\lambda_{ac}}, \frac{E_{bc}E_{cb}}{\lambda_{bc}}\right] - \left[\frac{E_{ac}E_{ca}}{\lambda_{ac}}, \frac{E_{ba}E_{ab}}{\lambda_{ba}}\right] =$$
$$= \frac{E_{ac}E_{ba}E_{cb} - E_{bc}E_{ab}E_{ba}}{\lambda_{ab}\lambda_{bc}} - \frac{E_{ac}E_{ba}E_{cb} - E_{bc}E_{ab}E_{ca}}{\lambda_{ac}\lambda_{bc}} + \frac{E_{ac}E_{ba}E_{cb} - E_{bc}E_{ab}E_{ca}}{\lambda_{ac}\lambda_{ba}} = 0.$$

The case of the equal parity of a, b and c is not to be considered because the corresponding $\mathfrak{gl}(n|m)$ operators are bosonic.

The calculations shows that the system of Dynamical equations is compatible, hence the first part of the claim is correct.

Now let us show that Knizhnik–Zamolodchikov equations and Dynamical equations are compatible. It boils down to the vanishing of the following commutator

$$\left[\kappa\partial_{z_i} - \sum_c \lambda_c e_{cc}^{(i)} - \sum_{j,\neq i} \frac{P_{ij}}{z_i - z_j}, \kappa\partial_a - \sum_j z_j e_{aa}^{(j)} - \sum_{b,\neq a} (-1)^{p(b)} \frac{E_{ab}E_{ba} - E_{aa}}{\lambda_a - \lambda_b}\right] = 0.$$

Let us take the most tricky part of the commutator above

1.
$$\left[\sum_c \lambda_c e_{cc}^{(i)}, \sum_{b,\neq a} (-1)^{p(b)} \frac{E_{ab}E_{ba} - E_{aa}}{\lambda_{ab}}\right] + \left[\sum_{j,\neq i} \frac{P_{ij}}{z_{ij}}, \sum_j z_j e_{aa}^{(j)}\right] =$$
$$= \sum_{b,\neq a} (-1)^{p(b)} \left(e_{ab}^{(i)} E_{ba} - E_{ab} e_{ba}^{(i)} - e_{ab}^{(i)} E_{ba} + E_{ab} e_{ba}^{(i)}\right) = 0.$$

2.
$$\left[\sum_{j,\neq i} \frac{P_{ij}}{z_i - z_j}, \sum_{b,\neq a} (-1)^{p(b)} \frac{E_{ab}E_{ba} - E_{aa}}{\lambda_a - \lambda_b}\right] = 0.$$

A simple calculation shows that the equality below is correct

$$[P_{ij}, E_{ab}E_{ba}] = 0.$$

The Calogero–Moser model.

Let us compute

$$\begin{aligned} \langle\Omega^i|\sum_{a=1}^n D_a^2|\Psi\rangle = \langle\Omega|\Bigg(&\sum_{a=1}^n \kappa\frac{\partial^2}{\partial\lambda_a^2} - \{\sum_{i,a=1}^n z_i e_{aa}^{(i)}, \sum_{b\neq a} (-1)^{p(b)} \frac{E_{ab}E_{ba} - E_{aa}}{\lambda_a - \lambda_b}\} - \\ &- \left(\sum_i z_i e_{aa}^{(i)}\right)^2 - \sum_{b\neq c\neq a} \frac{(-1)^{p(b)+p(c)}(E_{ab}E_{ba} - E_{aa})(E_{ac}E_{ca} - E_{aa})}{(\lambda_a - \lambda_b)(\lambda_a - \lambda_c)} + \\ &+ \sum_{b\neq a} \frac{\kappa(-1)^{p(b)}(E_{ab}E_{ba} - E_{aa}) + (E_{ab}E_{ba} - E_{aa})^2}{(\lambda_a - \lambda_b)^2}\Bigg)|\Psi\rangle, \end{aligned} \tag{S2}$$

during the transition from left to right of (S2) the Dynamical equations were used. The above expression simplifies to

$$\langle\Omega^i|(\sum_{a=1}^{n+m} \kappa^2 \frac{\partial^2}{\partial\lambda_a^2} - \{\sum_{i,a=1}^{n+m} z_i e_{aa}^{(i)}, \sum_{b\neq a} (-1)^{p(b)} \frac{E_{ab}E_{ba} - E_{aa}}{\lambda_a - \lambda_b}\} - \sum_i z_i^2 + \sum_{b\neq a} \frac{\kappa(-1)^i - 1}{(\lambda_a - \lambda_b)^2})|\Psi\rangle = 0.$$

To prove the claim one has to show the following

$$\langle\Omega^i|\left(\{\sum_{i,a=1}^{n+m} z_i e_{aa}^{(i)}, \sum_{b\neq a}(-1)^{p(b)}\frac{E_{ab}E_{ba}-E_{aa}}{\lambda_a-\lambda_b}\}\right)|\Psi\rangle = 0.$$

It is necessary to check that for any distinct a and b

$$\langle\Omega^i|\left(\{\sum_{i=1}^{n+m} z_i e_{aa}^{(i)}, (-1)^{p(b)}(E_{ab}E_{ba}-E_{aa})\} - \{\sum_{i=1}^{n+m} z_i e_{bb}^{(i)}, (-1)^{p(a)}(E_{ba}E_{ab}-E_{bb})\}\right) = 0.$$

After opening the brackets one gets the following

$$\langle\Omega^i|\left((\sum_{i=1}^{n} z_i e_{aa}^{(i)})(-1)^{p(b)}(E_{ab}E_{ba}-E_{aa})\right) + (-1)^i\langle\Omega^i|(\sum_{i=1}^{n} z_i e_{aa}^{(i)}) -$$
$$- \langle\Omega^i|\left((\sum_{i=1}^{n} z_i e_{bb}^{(i)})(-1)^{p(a)}(E_{ba}E_{ab}-E_{bb})\right) - (-1)^i\langle\Omega^i|(\sum_{i=1}^{n} z_i e_{bb}^{(i)}).$$

One can present the projectors in the following way

$$\langle\Omega^i| = \sum_{a_1\neq\ldots\neq a_{n+m}} \langle e_{a_1}\otimes\ldots\otimes e_a^{(i)}\otimes\ldots\otimes e_b^{(j)}\otimes\ldots\otimes e_{a_{n+m})}|f_i(a_1,\ldots,a_{n+m}).$$

The functions f_i generalize the sign function of the ordinary permutation to the graded case.

Let $i=0$ and take $p(a)=0$, $p(b)=1$ (the most non-trivial case) explicitly

$$\sum_{i=1}^{n}\left(\sum_{j,\neq i}\sum_{\{a_i=a,a_j=b\}}\langle e_{a_1}\otimes\ldots\otimes e_a^{(i)}\otimes\ldots\otimes e_b^{(j)}\otimes\ldots\otimes e_{a_{n+m})}|f_0(a_1,\ldots,a_{n+m})\right)\times$$
$$\times z_i\left((-1)^{p(b)}(E_{ab}E_{ba}-E_{aa})+1\right) =$$
$$= \sum_{i=1}^{n+m}\left(\sum_{j,\neq i}\sum_{\{a_i=a,a_j=b\}}\Big[\langle e_{a_1}\otimes\ldots\otimes e_b^{(i)}\otimes\ldots\otimes e_a^{(j)}\otimes\ldots\otimes e_{a_{m+N}}|(-1)^{\sum_{k\in\{i,j\}}p(a_k)} +\right.$$
$$\left.+\langle e_{\sigma(1)}\otimes\ldots\otimes e_a^{(i)}\otimes\ldots\otimes e_b^{(j)}\otimes\ldots\otimes e_{a_{n+m}}|\Big]f_0(a_1,\ldots,a,\ldots,b,\ldots,a_{n+m})\right)z_i,$$

where $\sum_{\{a_i=a,a_j=b\}} = \sum_{a_1\neq\ldots\neq a\neq\ldots\neq b\neq\ldots\neq a_{m+n}}$ and $\sum_{k\in\{i,j\}}$ means that the summation runs through all k between i and j. So the second line gives the following

$$\sum_{i=1}^{n+m}\left(\sum_{j,\neq i}\sum_{\{a_i=b,a_j=a\}}\Big[\langle e_{a_1}\otimes\ldots\otimes e_a^{(i)}\otimes\ldots\otimes e_b^{(j)}\otimes\ldots\otimes e_{a_{m+N}}|(-1)^{\sum_{k\in\{i,j\}}p(a_k)}+\right.$$
$$\left.+\langle e_{\sigma(1)}\otimes\ldots\otimes e_b^{(i)}\otimes\ldots\otimes e_a^{(l)}\otimes\ldots\otimes e_{a_{n+m}}|\Big]f_0(a_1,\ldots,b,\ldots,a,\ldots,a_{n+m})\right)z_i.$$

So one has the following relation

$$f_0(a_1,\ldots,a,\ldots,b,\ldots,a_{n+m}) = (-1)^{\sum_{k\in\{i,j\}}p(a_k)}f_0(a_1,\ldots,b,\ldots,a,\ldots,a_{n+m}).$$

One sees that the difference vanishes. Consideration of other parities of a, b as well as other values of i is completely analogous.