Supplementary Material to the article "Nonlinear kinetic inductance sensor"

To calculate dependence $I_{SN}(q_{SN})$ of SN bilayer we use the one-dimensional Usadel equation for normal g and anomalous f quasi-classical Green functions. With standard angle parametrization $g = cos\Theta$ and $f = sin\Theta\exp(i\varphi)$ the Usadel equations in different layers can be written as

$$\frac{\hbar D_S}{2}\frac{\partial^2\Theta_S}{\partial x^2} - \left(\hbar\omega_n + \frac{D_S}{2\hbar}q_{SN}^2\cos\Theta_S\right)\sin\Theta_S + \Delta\cos\Theta_S = 0, \tag{1}$$

$$\frac{\hbar D_N}{2}\frac{\partial^2\Theta_N}{\partial x^2} - \left(\hbar\omega_n + \frac{D_N}{2\hbar}q_{SN}^2\cos\Theta_N\right)\sin\Theta_N = 0, \tag{2}$$

where D is the diffusion coefficient for corresponding layer, $\hbar\omega_n = \pi k_B T(2n+1)$ are the Matsubara frequencies (n is an integer number), $\hbar q_{SN} = \hbar(\nabla\varphi + 2\pi\,\mathbf{A}/\Phi_0)$ is the momentum of Cooper pairs, φ is the phase of the order parameter, $\mathbf{A}$ is the vector potential, $\Phi_0 = \pi\hbar c/|e|$ is the magnetic flux quantum. Δ is the superconducting order parameter, which satisfies to the self-consistency equation

$$\Delta\ln\left(\frac{T}{T_{c0}}\right) = 2\pi k_B T\sum_{\omega_n>0}\left(\sin\Theta_S - \frac{\Delta}{\hbar\omega_n}\right), \tag{3}$$

where T_{c0} is the critical temperature of single S layer in the absence of magnetic field. These equations are supplemented by the Kupriyanov-Lukichev boundary conditions on SN interface [1]

$$D_S\frac{d\Theta_S}{dx} = D_N\frac{d\Theta_N}{dx} \tag{4}$$

which corresponds to the case with zero barrier between layers and continuous Θ on SN interface. For interfaces with vacuum we use the boundary condition $d\Theta/dx = 0$.

We assume that the thickness $d_S + d_N$ of SN strip is much smaller than the London penetration depth λ while the width w is smaller than the Pearl penetration depth $\Lambda = \lambda^2/(d_S+d_N)$ which allows us to neglect the magnetic field created by supercurrent and put $A = 0$.

To calculate the absolute value of current density j and current $I_{SN} = w\int jdx$ we use the following expression

$$j(x) = \frac{2\pi k_B T}{|e|\rho}q_{SN}\sum_{\omega_n>0}\sin^2\Theta \tag{5}$$

where $\rho = 2|e|D_{S,N}N(0)$ is the residual resistivity of the corresponding layer, $N(0)$ is density of states of electrons per one spin at the Fermi level in the normal state (for simplicity we assume identical $N(0)$ in S and N layers).

Equations (1-3) are solved numerically by using iteration procedure. For initial distribution $\Delta(x) = const$ and chosen q_{SN} we solve Eqs. (1,2) (in numerical procedure we use Newton method combined with tridiagonal matrix algorithm). Found solution $\Theta(x)$ is inserted to Eq. (3) to find $\Delta(x)$ and than iterations repeat until the relative change in $\Delta(x)$ between two iterations does not exceed 10^{-8}. Length is normalized in units of $\xi_c = \sqrt{\hbar D_S/k_B T_{c0}}$, energy is in units of $k_B T_{c0}$, current is in units of depairing current $I_{dep,S}$ of single S layer with the thickness d_S. Typical step grid in S and N layers is $\delta x = 0.1\xi_c$.

With calculated $I_{SN}(q_{SN})$ we find kinetic inductance per unit of length of the strip

$$L_k = \hbar c^2(dI_{SN}/dq_{SN})^{-1}/2|e|. \tag{6}$$

To find the local density of states, we do the analytic continuation from the Matsubara frequencies to the quasiparticle energies $\hbar\omega_n \to -iE$ in Eqs. (1,2) and use the expression $N(x,E) = N(0)Re(\cos\theta(x,E))$. To decrease time of calculations we use dependence $\Delta(x)$ found previously in Matsubara representation.

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[1] M. Yu. Kuprianov and V. F. Lukichev, Influence of boundary transparency on the critical current of "dirty" SS'S structures, Sov. Phys. JETP **67**, 1163 (1988).