Supplementary Material to the article

"Controlling the dynamic modes of a nanoscale superconducting interferometer"

In this supplementary material we derive the solution of the second order inhomogeneous differential equation:

$$\ddot{\varphi} + \alpha\dot{\varphi} + (1 + \varepsilon\, Cos(2\omega_0\tau + \Delta))\omega_0^2\varphi = i_s\, Cos(\omega_0\tau). \quad (S1)$$

The general solution of the equation (S1) is the sum of the general solution of a homogeneous equation and the particular solution of an inhomogeneous equation:

$$\varphi(\tau) = \varphi_{gshe}(\tau) + \varphi_{psie}(\tau). \quad (S2)$$

We assume the general solution of the homogeneous equation as $\varphi_{gshe}(\tau) = \mu e^{k\tau} Cos(\omega_0\tau) + \nu e^{k\tau} Sin(\omega_0\tau)$. After the substituting of $\varphi_{gshe}(\tau)$ into the equation (S1) the following coupled linear equations for the coefficients μ and ν were obtained:

$$\nu\left(2\omega_0 k + \alpha\omega_0 - \frac{\omega_0^2\varepsilon}{2} Sin(\Delta)\right) + \mu\left(k\alpha + \frac{\omega_0^2\varepsilon}{2} Cos(\Delta)\right) = 0, \quad (S3a)$$

$$\mu\left(-2\omega_0 k - \alpha\omega_0 - \frac{\omega_0^2\varepsilon}{2} Sin(\Delta)\right) + \nu\left(k\alpha - \frac{\omega_0^2\varepsilon}{2} Cos(\Delta)\right) = 0. \quad (S3b)$$

The compatibility of the solution of these equations leads to specific values of exponential coefficients k in the following form:

$$k_1 = \frac{(-4\omega_0{}^2\alpha^4 + \varepsilon^2\alpha^2\omega_0{}^4 + 4\varepsilon^2\omega_0^6)^{1/2}}{2(\alpha^2 + 4\omega_0^2)} - \frac{4\omega_0{}^2\alpha}{2(\alpha^2 + 4\omega_0^2)}, \quad (S4)$$

$$k_2 = -\frac{(-4\omega_0{}^2\alpha^4 + \varepsilon^2\alpha^2\omega_0{}^4 + 4\varepsilon^2\omega_0^6)^{\frac{1}{2}}}{2(\alpha^2 + 4\omega_0^2)} - \frac{4\omega_0{}^2\alpha}{2(\alpha^2 + 4\omega_0^2)}. \quad (S5)$$

As a result:

$$\varphi_{gshe}(\tau) = (\mu_1 e^{k_1\tau} + \mu_2 e^{k_2\tau}) Cos(\omega_0\tau) + (\eta_1\mu_1 e^{k_1\tau} + \eta_2\mu_2 e^{k_2\tau}) Sin(\omega_0\tau) \quad (S6)$$

where

$$\nu_i = -\frac{\alpha k_i + \omega_0^2\frac{\varepsilon}{2}Cos(\Delta)}{2\omega_0 k_i + \omega_0\alpha - \omega_0^2\frac{\varepsilon}{2}Sin(\Delta)}\,\mu_i = \eta_i\mu_i,\ i = 1{,}2\,. \quad (S7)$$

We assume the particular solution of the inhomogeneous equation (S1) as $\varphi_{psie}(\tau) = ACos(\omega_0\tau) + BSin(\omega_0\tau)$. The substitution of $\varphi_{psie}(\tau)$ into the equation (S1) leads to the following coupled linear equations for the coefficients A и B:

$$\omega_0\alpha B + \frac{\omega_0^2\varepsilon}{2} ACos(\Delta) - \frac{\omega_0^2\varepsilon}{2} BSin(\Delta) = i_s, \quad (S8a)$$

$$-\omega_0\alpha A - \frac{\omega_0^2\varepsilon}{2} BCos(\Delta) - \frac{\omega_0^2\varepsilon}{2} ASin(\Delta) = 0. \quad (S8b)$$

We thus have:

$$A = \frac{2\, i_s\varepsilon\, Cos(\Delta)}{\omega_0{}^2\varepsilon^2 - 4\alpha^2}, \quad (S9)$$

$$\mathrm{B} = -\frac{4 i_s\alpha}{(\omega_0{}^2\varepsilon^2 - 4\alpha^2)\omega_0} - \frac{2\,\varepsilon\, i_s Sin(\Delta)}{(\omega_0{}^2\varepsilon^2 - 4\alpha^2)}. \quad (S10)$$

Next, we apply the initial conditions in order to determine the coefficients μ_1 and μ_2:

$$\mu_1 + \mu_2 + A = i_s, \quad (S11a)$$

$$\omega_0(\eta_1\mu_1 + \eta_2\mu_2) + \mathrm{B}\omega_0 + k_1\mu_1 + k_2\mu_2 = 0. \quad (S11b)$$

Solving equations (S11), we find μ_1 and μ_2 to be:

$$\mu_1 = -\frac{(i_s - A)(\omega_0\eta_2 + k_2) + B\omega_0}{k_1 - k_2 + \eta_1\,\omega_0 - \eta_2\,\omega_0}, \quad (S12)$$

$$\mu_2 = -\frac{(A - i_s)(\omega_0\eta_1 + k_1) - B\omega_0}{k_1 - k_2 + \eta_1\,\omega_0 - \eta_2\,\omega_0}. \quad (S13)$$

Thus, the total solution of the equation (S1) takes the following form:

$$\varphi(\tau) = Cos(\omega_0\tau)(A + \mu_1 e^{k_1\tau} + \mu_2 e^{k_2\tau}) + Sin(\omega_0\tau)(B + \eta_1\mu_1 e^{k_1\tau} + \eta_2\mu_2 e^{k_2\tau}). \quad (S14)$$

A non-resonant case.

Similarly, to the approach given in the resonant case, we obtain an analytical solution of a second order inhomogeneous differential equation in the non-resonant case:

$$\ddot{\varphi} + \alpha\dot{\varphi} + [1 + \varepsilon\, Cos(2\Omega\tau + \Delta)]\omega_0^2\varphi = i_s Cos(\Omega\tau). \quad (S15)$$

The general solution of the equation (S15) is the sum (S2) of the general solution of a homogeneous equation and the particular solution of an inhomogeneous equation. We assume the general solution of the homogeneous equation as $\varphi_{gshe}(\tau) = \delta e^{k\tau} Cos(\Omega\tau) + \rho e^{k\tau} Sin(\Omega\tau)$. The substitution of $\varphi_{gshe}(\tau)$ into the equation (S15) leads to the following coupled linear equations for the coefficients δ и ρ:

$$\delta\left(\alpha\gamma + \omega_0^2 - \Omega^2 + \frac{\varepsilon\omega_0^2}{2} Cos(\Delta)\right) + \beta\left(2\Omega\gamma + \Omega\alpha - \frac{\varepsilon\omega_0^2}{2} Sin(\Delta)\right) = 0, \quad (S16a)$$

$$\beta\left(-2\Omega\gamma + \omega_0^2 - \Omega^2 - \frac{\varepsilon\omega_0^2}{2} Cos(\Delta)\right) + \delta\left(\alpha\gamma - \Omega\alpha - \frac{\varepsilon\omega_0^2}{2} Sin(\Delta)\right) = 0. \quad (S16b)$$

The compatibility of the solution of these equations leads to specific values of exponential coefficients γ in the following form:

$$\gamma_1 = \frac{1}{\alpha^2 + 4\Omega^2}\left(\alpha^2(\omega_0^2 + \Omega^2)^2 - (\alpha^2 + 4\Omega^2)(\alpha^2\Omega^2 + \Omega^4 - 2\omega_0^2\Omega^2 + \omega_0^4 - (\frac{\varepsilon\omega_0^2}{2})^2)\right)^{1/2} - \frac{\alpha(\omega_0^2 + \Omega^2)}{\alpha^2 + 4\Omega^2}, \quad (S17)$$

$$\gamma_2 = -\frac{1}{\alpha^2 + 4\Omega^2}\left(\alpha^2(\omega_0^2 + \Omega^2)^2 - (\alpha^2 + 4\Omega^2)(\alpha^2\Omega^2 + \Omega^4 - 2\omega_0^2\Omega^2 + \omega_0^4 - (\frac{\varepsilon\omega_0^2}{2})^2)\right)^{1/2} - \frac{\alpha(\omega_0^2 + \Omega^2)}{\alpha^2 + 4\Omega^2}. \quad (S18)$$

In that way:

$$\varphi_{\text{opoy}}[\tau] = (\delta_1 e^{\gamma_1\tau} + \delta_2 e^{\gamma_2\tau}) Cos(\Omega\tau) + (\chi_1\delta_1 e^{\gamma_1\tau} + \chi_2\delta_2 e^{\gamma_2\tau}) Sin(\Omega\tau) \quad (S19)$$

where

$$\rho_i = -\frac{\gamma_i\alpha + \omega_0^2 - \Omega^2 + \frac{\varepsilon\omega_0^2}{2} Cos(\Delta)}{2\Omega\gamma_i + \alpha\Omega - \frac{\varepsilon\omega_0^2}{2} Sin(\Delta)}\ \chi_i = \chi_i\delta_i, \quad i = 1,2\ . \quad (S20)$$

We assume the particular solution of the inhomogeneous equation (S1) as $\varphi_{psie}(\tau) = A_1 e^{i\Omega t} + B_1 e^{-i\Omega t}$. After the substituting of $\varphi_{gshe}(\tau)$ into the equation (S15) the following coupled linear equations for the coefficients A_1 and B_1 were obtained:

$$A_1(\omega_0^2 - \Omega^2 + i\alpha\Omega) + \frac{\varepsilon\omega_0^2}{2}\big(Cos(\Delta) + iSin(\Delta)\big)B_1 = \frac{i_s}{2}, \quad (S21a)$$

$$B_1(\omega_0^2 - \Omega^2 - i\alpha\Omega) + \frac{\varepsilon\omega_0^2}{2}\big(Cos(\Delta) - iSin(\Delta)\big)A_1 = \frac{i_s}{2}. \quad (S21b)$$

We thus have

$$A_1 = B_1^+ = i_s\frac{\omega_0^2\varepsilon Cos(\Delta) - 2(\omega_0^2 - \Omega^2)}{(\omega_0^2\varepsilon)^2 - 4(\omega_0^2 - \Omega^2)^2 + (\alpha\Omega)^2)} + i_s\frac{2\alpha\Omega + \omega_0^2\varepsilon Sin(\Delta)}{(\omega_0^2\varepsilon)^2 - 4((\omega_0^2 - \Omega^2)^2 + (\alpha\Omega)^2)} i = \tilde{A}_1 + \tilde{B}_1 i. \quad (S22)$$

It is more convenient to introduce new variables defined by $A_0 = 2\tilde{A}_1$ and $B_0 = -2\tilde{B}_1$. In that case:

$$\varphi_{psie}(\tau) = A_0 Cos(\Omega\,\tau) + B_0\, Sin(\Omega\,\tau). \quad (S23)$$

Next, we use the initial conditions in order to determine the coefficients δ_1 and δ_2:

$$\delta_1 + \delta_2 + A_0 \ = i_s, \qquad (S24a)$$

$$\Omega(\chi_1\delta_1 + \chi_2\delta_2) + B_0\Omega + \gamma_1\delta_1 + \gamma_2\delta_2 = 0. \qquad (S24b)$$

from where follows:

$$\delta_1 = -\frac{(\ i_s - A_0)(\omega_0\chi_2 + \gamma_2) - B_0\Omega}{\gamma_1 - \gamma_2 + \chi_1\ \omega_0 - \chi_2\ \omega_0}, \qquad (S25)$$

$$\delta_2 = \frac{(\ i_s - A_0)(\omega_0\chi_1 + \gamma_1) - B_0\Omega}{\gamma_1 - \gamma_2 + \chi_1\ \omega_0 - \chi_2\ \omega_0}. \qquad (S26)$$

Thus, we obtained an analytical solution of the equation (S15):

$$\begin{aligned}\varphi(\tau) = (\delta_1 Exp(\gamma_1\tau) + \delta_2 Exp(\gamma_2\tau) + A_0)Cos(\Omega\,\tau) \\ + (\delta_1\chi_1 Exp(\gamma_1\tau) + \delta_2\chi_2 Exp(\gamma_2\tau) \\ + B_0)Sin(\Omega\,\tau). \qquad (S27)\end{aligned}$$