Supplementary Material to the article "Anomalous Josephson effect in a planar hybrid structure with a spin–orbit interaction"

A. Solution of Poisson's equation

The spatial distribution of the phase $\phi(\mathbf{r})$ of the order parameter in thin superconducting film is described by the 2D Poisson's equation

$$\triangle\,\phi(\mathbf{r}) = \mathrm{div}\,\mathbf{S}(\mathbf{r})\,, \tag{S1}$$

where the source $\mathbf{S}(\mathbf{r}) = -\alpha(\mathbf{r})\,\mathbf{e}_\alpha$ in the right side (S1) is non-zero in the area under the ferromagnetic disk ($r = \sqrt{x^2+y^2} \le R$). Here the unit vector $\mathbf{e}_\alpha = [\mathbf{e}_h,\,\mathbf{z}_0] = \mathbf{x}_0 \sin\chi - \mathbf{y}_0\cos\chi$ is determined by the direction of the exchange field $\mathbf{e}_h = \mathbf{x}_0\cos\chi + \mathbf{y}_0\sin\chi$. The solution to the equation (S1) is obtained using the Fourier transforms

$$\phi(\mathbf{r}) = \frac{1}{(2\pi)^2}\int d\mathbf{q}\,\phi(\mathbf{q})\,e^{-i\mathbf{q}\,\mathbf{r}}\,, \qquad \alpha(\mathbf{r}) = \frac{1}{(2\pi)^2}\int d\mathbf{q}\,\alpha(\mathbf{q})\,e^{-i\mathbf{q}\,\mathbf{r}}\,,$$

where $\mathbf{q} = q\,(\mathbf{x}_0\cos\vartheta + \mathbf{y}_0\sin\vartheta)$ is the wavevector in the plane of the film. The amplitude of the Fourier harmonic of the parameter $\alpha(\mathbf{r})$, which is responsible for the emergence of inhomogeneous phase distribution and spontaneous supercurrent in the film can be expresses in terms of Bessel unction $J_1(u)$ of the first kind:

$$\alpha(\mathbf{q}) = \int\limits_{r\le R} d\mathbf{r}\,\alpha(\mathbf{r})\,e^{i\mathbf{q}\,\mathbf{r}} = 2\pi R\alpha_0\frac{J_1(qR)}{q}\,.$$

As a result, the following expression for the amplitude of Fourier harmonics the phase distribution takes place

$$\phi(\mathbf{q}) = -\frac{i\,\alpha(\mathbf{q})}{q^2}\,(\mathbf{q},\,\mathbf{e}_\alpha\,) = -2\pi i\,R\alpha_0\frac{J_1(qR)}{q^3}\,(\mathbf{q},\,\mathbf{e}_\alpha)\;. \tag{S2}$$

Using relation (S2) and Fourier transforms, we obtain the following expressions for the distributions of the order parameter phase in the film

$$\begin{aligned}
\phi(\mathbf{r}) &= \frac{1}{(2\pi)^2}\int\limits_0^\infty q\,dq\int\limits_0^{2\pi} d\vartheta\,\frac{2i\pi R\,\alpha_0 J_1(qR)}{q^2}\,\left(\sin\vartheta\cos\chi - \cos\vartheta\sin\chi\,\right)\,e^{-iqr\cos(\vartheta-\theta)} = \\
&= \frac{iR\,\alpha_0}{2\pi}\int\limits_0^\infty dq\,\frac{J_1(qR)}{q}\left[\cos(\theta-\chi)\int\limits_0^{2\pi} d\beta\sin\beta\,e^{-iqr\cos\beta} + \sin(\theta-\chi)\int\limits_0^{2\pi} d\beta\cos\beta\,e^{-iqr\cos\beta}\right] = \\
&= R\,\alpha_0\,\sin(\theta-\chi)\int\limits_0^\infty dq\,\frac{J_1(qR)\,J_1(qr)}{q} = R\,\alpha_0\,\left[\sin\theta\,\cos\chi - \cos\theta\,\sin\chi\right])\left\{\begin{array}{l} r/2R,\;\; r\le R \\ R/2r,\;\; r>R \end{array}\right. = \\
&= \frac{\alpha_0}{2}\left\{\begin{array}{ll} y\,\cos\chi - x\,\sin\chi\,, & r\le R \\ \dfrac{R^2}{r^2}\,\left(\,y\,\cos\chi - x\,\sin\chi\,\right)\,, & r>R \end{array}\right.\,,
\end{aligned} \tag{S3}$$

where $\mathbf{r} = r\,(\mathbf{x}_0\cos\theta + \mathbf{y}_0\sin\theta)$. Using relation (S3) one can obtain the following expressions for the distributions of the spontaneous current in a thin superconducting film near the ferromagnetic disk

$$\mathbf{g}(\mathbf{r}) = (\,\mathrm{g}_x(\mathbf{r}),\,\mathrm{g}_y(\mathbf{r})\,) = -\frac{c\,\Phi_0}{8\pi^2\Lambda}\,(\,\nabla\phi + \alpha(\mathbf{r})\,) \tag{S4}$$

$$\mathrm{g}_x(\mathbf{r}) = -\frac{c\,\Phi_0\,\alpha_0}{16\pi^2\Lambda}\left\{\begin{array}{ll} \sin\chi, & r<R \\ -\dfrac{R^2}{r^2}\,\sin(2\theta-\chi), & r>R \end{array}\right.\,, \qquad \mathrm{g}_y(\mathbf{r}) = \frac{c\,\Phi_0\,\alpha_0}{16\pi^2\Lambda}\left\{\begin{array}{ll} \cos\chi, & r<R \\ -\dfrac{R^2}{r^2}\,\cos(2\theta-\chi), & r>R \end{array}\right. \tag{S5}$$

B. Average phase shift

The phase shift φ_0 between the electrodes away from the junction area can be estimated as

$$\varphi_0 \approx -\overline{\varphi} = -\int_0^1 dy\, \varphi(y)\,, \tag{S6}$$

where

$$\varphi(y) = \frac{\alpha_0 R^2}{W}\, \frac{x_d\, \sin\chi + (y - y_d)\, \cos\chi}{x_d^2 + (y - y_d)^2} + \psi_0(y)\,, \tag{S7}$$

$$\psi_0(y) = \frac{\alpha_0 R^2}{4\pi W} \left\{ \cos\chi \int_{-\infty}^{+\infty} du\, A(u)\, [Q_w(0,y,u) - Q_d(0,y,u)] + \sin\chi \int_{-\infty}^{+\infty} du\, B(u)\ [Q_d(0,y,u) + Q_w(0,y,u)] \right\}, \tag{S8}$$

$$A(u) = \frac{(u - x_d)^2 - y_d^2}{[\,(u - x_d)^2 + y_d^2\,]^2} + \frac{(u + x_d)^2 - y_d^2}{[\,(u + x_d)^2 + y_d^2\,]^2}\,, \quad B(u) = \left[\frac{2y_d\,(u - x_d)}{[\,(u - x_d)^2 + y_d^2\,]^2} - \frac{2y_d\,(u + x_d)}{[(u + x_d)^2 + y_d^2\,]^2} \right],$$

$$Q_{d(w)}(x, y, u) = \ln\left[\cosh\left(\pi \frac{x - u}{W} \right) \mp \cos\left(\frac{\pi y}{W} \right) \right].$$

$$\int_0^1 dy\, \frac{x_d\, \sin\chi + (y - y_d)\, \cos\chi}{x_d^2 + (y - y_d)^2} = 2\, \sin\chi\, \mathrm{arctg}\left(\frac{1}{2x_d} \right) \tag{S9}$$

$$\overline{\psi_0} = \int_0^1 dy\, \psi_0(y) = \frac{\alpha_0 R^2}{4\pi W} \left\{ \cos\chi \int_{-\infty}^{+\infty} du\, A(u)\, \left[\overline{Q}_w(u) - \overline{Q}_d(u)\right] + \sin\chi \int_{-\infty}^{+\infty} du\, B(u)\ \left[\overline{Q}_d(u) + \overline{Q}_w(u)\right] \right\}, \tag{S10}$$

$$\overline{Q}_d = \overline{Q}_w = \int_0^1 dy\, \ln\left[\, \cosh\left(\pi u\right) \mp \cos\left(\pi y\right) \right] = \pi|u| - \ln 2$$

$$\overline{\psi_0} = \frac{2\alpha_0\, R^2}{W}\, \sin\chi\, \mathrm{arctg}\left(2x_d\right) \tag{S11}$$

$$\varphi_0 \approx -\overline{\varphi} = -\frac{2\alpha_0\, R^2}{W}\, \sin\chi\, \left[\mathrm{arctg}\left(\frac{1}{2x_d} \right) + \mathrm{arctg}\left(2x_d\right) \right] = -\frac{\alpha_0\, R^2}{W}\, \sin\chi \tag{S12}$$