

# Supplementary Material to the article ‘‘Generalized model of the superconducting sigma-neuron’’

## I. THE GENERAL INVERSE MATRIX COMPONENTS OF A SIGMA NEURON EQUATION OF STATE

The inverse matrix elements can be expressed as  $A_{pq}^{-1} = (-1)^{p+q} \Delta_{qp} / \Delta$ , where  $\Delta$  is the determinant of the matrix  $\|A\|$ , and  $\Delta_{qp}$  is a minor which is a determinant of a matrix obtained from  $\|A\|$  by deleting the  $q$ -th row and the  $p$ -th column (see e.g. [1]). In the general case, the determinants are polynomials of the second (for  $\Delta_{qp}$  with  $q = 1, 2$ , or  $3$ ) or third (for  $\Delta$  and  $\Delta_{qp}$  with  $q = 4$  or  $5$ ) degree in the inductance matrix components. The ones needed for calculating the transfer function coefficients are written out below for reference purposes.

$$\begin{aligned} \Delta = & -l_{14}l_{25}^2 - l_{15}l_{23}l_{25} + l_{15}l_{24}l_{25} + l_{25}^2l_{13} - l_{15}l_{25}l_{34} + l_{15}l_{22}l_{35} + 2l_{15}l_{24}l_{35} - l_{12}l_{25}l_{35} - l_{14}l_{25}l_{35} + l_{15}l_{25}l_{44} \\ & + l_{15}l_{35}l_{44} + l_{12}l_{45}^2 + l_{13}l_{45}^2 - l_{15}l_{22}l_{45} - l_{15}l_{23}l_{45} - l_{15}l_{24}l_{45} + l_{12}l_{25}l_{45} - l_{14}l_{25}l_{45} + 2l_{25}l_{13}l_{45} - l_{15}l_{34}l_{45} \\ & - l_{12}l_{35}l_{45} - l_{14}l_{35}l_{45} + l_{14}l_{22}l_{55} + l_{12}l_{23}l_{55} + l_{14}l_{23}l_{55} - l_{12}l_{24}l_{55} + l_{14}l_{24}l_{55} - l_{22}l_{13}l_{55} - 2l_{24}l_{13}l_{55} + l_{12}l_{34}l_{55} \\ & + l_{14}l_{34}l_{55} - l_{12}l_{44}l_{55} - l_{13}l_{44}l_{55}, \end{aligned}$$

$$\Delta_{11} = l_{25}^2 + l_{45}^2 + 2l_{25}l_{45} - l_{22}l_{55} - 2l_{24}l_{55} - l_{44}l_{55},$$

$$\begin{aligned} \Delta_{51} = & -l_{25}^2l_{33} + 2l_{25}^2l_{34} - l_{22}l_{35}^2 - 2l_{24}l_{35}^2 + 2l_{23}l_{25}l_{35} - 2l_{24}l_{25}l_{35} + 2l_{25}l_{34}l_{35} - l_{25}^2l_{44} - l_{35}^2l_{44} - 2l_{25}l_{35}l_{44} \\ & - l_{22}l_{45}^2 - 2l_{23}l_{45}^2 - l_{33}l_{45}^2 - 2l_{23}l_{25}l_{45} + 2l_{24}l_{25}l_{45} - 2l_{25}l_{33}l_{45} + 2l_{25}l_{34}l_{45} + 2l_{22}l_{35}l_{45} + 2l_{23}l_{35}l_{45} + 2l_{24}l_{35}l_{45} \\ & + 2l_{34}l_{35}l_{45} - l_{23}^2l_{55} - l_{24}^2l_{55} + 2l_{23}l_{24}l_{55} + l_{22}l_{33}l_{55} + 2l_{24}l_{33}l_{55} - l_{34}^2l_{55} - 2l_{22}l_{34}l_{55} - 2l_{23}l_{34}l_{55} - 2l_{24}l_{34}l_{55} \\ & + l_{22}l_{44}l_{55} + 2l_{23}l_{44}l_{55} + l_{33}l_{44}l_{55}, \end{aligned}$$

$$\Delta_{15} = l_{15}l_{22} + 2l_{15}l_{24} - l_{12}l_{25} - l_{14}l_{25} + l_{15}l_{44} - l_{12}l_{45} - l_{14}l_{45},$$

$$\begin{aligned} \Delta_{55} = & l_{15}l_{23}^2 + l_{15}l_{24}^2 - 2l_{15}l_{23}l_{24} + l_{14}l_{23}l_{25} - l_{14}l_{24}l_{25} - l_{23}l_{25}l_{13} + l_{24}l_{25}l_{13} - l_{15}l_{22}l_{33} - 2l_{15}l_{24}l_{33} + l_{12}l_{25}l_{33} \\ & + l_{14}l_{25}l_{33} + l_{15}l_{34}^2 + 2l_{15}l_{22}l_{34} + 2l_{15}l_{23}l_{34} + 2l_{15}l_{24}l_{34} - 2l_{12}l_{25}l_{34} - l_{14}l_{25}l_{34} - l_{25}l_{13}l_{34} - l_{14}l_{22}l_{35} - l_{12}l_{23}l_{35} \\ & - l_{14}l_{23}l_{35} + l_{12}l_{24}l_{35} - l_{14}l_{24}l_{35} + l_{22}l_{13}l_{35} + 2l_{24}l_{13}l_{35} - l_{12}l_{34}l_{35} - l_{14}l_{34}l_{35} - l_{15}l_{22}l_{44} - 2l_{15}l_{23}l_{44} + l_{12}l_{25}l_{44} \\ & + l_{25}l_{13}l_{44} - l_{15}l_{33}l_{44} + l_{12}l_{35}l_{44} + l_{13}l_{35}l_{44} + l_{14}l_{22}l_{45} + l_{12}l_{23}l_{45} + 2l_{14}l_{23}l_{45} - l_{12}l_{24}l_{45} - l_{22}l_{13}l_{45} - l_{23}l_{13}l_{45} \\ & - l_{24}l_{13}l_{45} + l_{12}l_{33}l_{45} + l_{14}l_{33}l_{45} - l_{12}l_{34}l_{45} - l_{13}l_{34}l_{45}. \end{aligned}$$

## II. SIGMA NEURON TRANSFER FUNCTION COEFFICIENTS IN THE FIRST CASE

In the first case (see Section 3.1 in the main text) the determinant of the matrix  $\|A\|$  contains 3 terms, which can be grouped as follows:

$$\Delta = -[l_{out}l_{sq} - m_{out}^2]m_{in} - \frac{m_{in}}{2}l_al_{sq} = -[l_{out}^*l_{sq}]m_{in} - \frac{m_{in}}{2}l_al_{sq} = -\frac{m_{in}}{2}(2l_{out}^* + l_a)l_{sq}, \quad (S1)$$

where  $l_{out}^* = l_{out} - m_{out}^2/l_{sq}$  is introduced. The minors  $\Delta_{11}$  and  $\Delta_{51}$  have 3 and 5 terms respectively. Using  $l_{out}^*$  one can write them in a compact form:

$$\Delta_{11} = -l_al_{sq} - [l_{out}l_{sq} - m_{out}^2] = -l_al_{sq} - [l_{out}^*l_{sq}] = -l_{sq}(l_a + l_{out}^*), \quad (S2)$$

$$\Delta_{51} = l_al_{sq} + (l + l_a)[l_{out}l_{sq} - m_{out}^2] = l_al_{sq} + (l + l_a)[l_{out}^*l_{sq}] = l_{sq}(l_al + l_{out}^*l + l_al_{out}^*) = l_{sq}(l(l_a + l_{out}^*) + l_al_{out}^*). \quad (S3)$$

The number of terms in the minors  $\Delta_{15}$  and  $\Delta_{55}$  is 1 and 2 respectively:

$$\Delta_{15} = -\frac{m_{in}}{2}m_{out}, \quad \Delta_{55} = -\frac{m_{in}}{2}m_{out}(l_a - l). \quad (S4)$$

Thus, one can obtain the expressions for the transfer function coefficients (see Eqs. (25)–(28) in Section 3.1 of the main text):

$$\alpha = -\frac{A_{15}^{-1}}{A_{11}^{-1}} = -\frac{\Delta_{51}}{\Delta_{11}} = -\frac{l(l_a + l_{out}^*) + l_a l_{out}^*}{-(l_a + l_{out}^*)} = l + \frac{l_a l_{out}^*}{(l_a + l_{out}^*)}, \quad (S5)$$

$$\sigma = -\frac{A_{55}^{-1}}{A_{51}^{-1}} = -\frac{\Delta_{55}}{\Delta_{15}} = -(l_a - l) = l - l_a, \quad (S6)$$

$$\frac{t_{cl}}{2\pi} = -A_{11}^{-1} = -\frac{\Delta_{11}}{\Delta} = -\frac{2}{m_{in}} \frac{(l_a + l_{out}^*)}{(l_a + 2l_{out}^*)}, \quad (S7)$$

$$\frac{t_{fb}}{2\pi} = -A_{51}^{-1} = -\frac{\Delta_{15}}{\Delta} = -\frac{m_{out}}{(2l_{out}^* + l_a)l_{sq}}. \quad (S8)$$

The negative sign of a period means that the shift by one period in the positive direction of current corresponds to a change in the Josephson phase  $\varphi$  by  $-2\pi$ .

### III. TRANSFER FUNCTION CONVERSION

Expression for the output current  $i_{fb}$  (see Eq. (17) from the main text) can be written in the same manner as in [2] provided that  $\varphi$  is expressed via the input current  $i_{cl}$  and the normalized Josephson current  $\sin \varphi$  using Eq. (16) from the main text:

$$\varphi = \frac{2\pi}{t_{cl}}(i_{cl} - \delta_{cl}) - \alpha \sin \varphi. \quad (S9)$$

Substituting Eq. (S9) into Eq. (17) from the main text, one obtains:

$$\begin{aligned} i_{fb} &= \frac{t_{fb}}{2\pi} \left( \frac{2\pi}{t_{cl}}(i_{cl} - \delta_{cl}) - \alpha \sin \varphi + \sigma \sin \varphi \right) + \delta_{fb} = \frac{t_{fb}}{2\pi} \left( \frac{2\pi}{t_{cl}}i_{cl} - (\alpha - \sigma) \sin \varphi + \frac{2\pi}{t_{fb}}\delta_{fb} - \frac{2\pi}{t_{cl}}\delta_{cl} \right) = \\ &= -\frac{t_{fb}(\alpha - \sigma)}{2\pi} \left( -\frac{2\pi}{t_{cl}(\alpha - \sigma)}i_{cl} - \sin(-\varphi) + \frac{2\pi}{(\alpha - \sigma)} \left[ \frac{\delta_{cl}}{t_{cl}} - \frac{\delta_{fb}}{t_{fb}} \right] \right). \end{aligned}$$

Thus, the coefficients  $k_3$ – $k_4$  have the form (33) from the main text.

Substituting Eqs. (18)–(19) from the main text into (32)–(33) ibidem, one can obtain expressions for the coefficients  $k_1$ – $k_4$  from [2]:

$$\begin{aligned} k_1 &= -\frac{t_{cl}}{2\pi} = \frac{2}{m_{in}} \frac{l_a + l_{out}^*}{l_a + 2l_{out}^*}, & k_2 &= \alpha = l + \frac{l_a l_{out}^*}{l_a + l_{out}^*}, \\ k_3 &= -\frac{t_{fb}(\alpha - \sigma)}{2\pi} = \frac{m_{out}}{(2l_{out}^* + l_a)l_{sq}} \left( l + \frac{l_a l_{out}^*}{(l_a + l_{out}^*)} - l + l_a \right) = \frac{m_{out}}{(2l_{out}^* + l_a)l_{sq}} \frac{l_a(2l_{out}^* + l_a)}{(l_a + l_{out}^*)} = \frac{m_{out}l_a}{l_{sq}(l_a + l_{out}^*)}, \\ k_4 &= -\frac{2\pi}{t_{cl}(\alpha - \sigma)} = \frac{m_{in}}{2} \frac{(l_a + 2l_{out}^*)}{(l_a + l_{out}^*)} \frac{(l_a + l_{out}^*)}{l_a(2l_{out}^* + l_a)} = \frac{m_{in}}{2l_a}. \end{aligned}$$

### IV. SIGMA NEURON TRANSFER FUNCTION COEFFICIENTS IN THE SECOND CASE

In the second case (see Section 3.2 in the main text) the number of terms in the expression for  $\Delta$  increases to 5, but they can be represented as

$$\Delta = -[l_{out}l_{sq} - m_{out}^2](m_a + m_J) - l_a l_{sq} m_J = -[l_{out}^*l_{sq}](m_a + m_J) - l_a l_{sq} m_J. \quad (S10)$$

The expressions for  $\Delta_{11}$  and  $\Delta_{51}$  remain the same (see Eqs. (S2)–(S3)), while  $\Delta_{15}$  and  $\Delta_{55}$  read:

$$\Delta_{15} = -m_a m_{out}, \quad \Delta_{55} = (l m_a - l_a m_J) m_{out}. \quad (\text{S11})$$

Thus, in the second case the transfer function coefficients will be (see Eqs. (34)–(37) in Section 3.2 of the main text):

$$\frac{t_{cl}}{2\pi} = -A_{11}^{-1} = -\frac{\Delta_{11}}{\Delta} = -\frac{l_a + l_{out}^*}{m_J l_a + (m_a + m_J) l_{out}^*}, \quad (\text{S12})$$

$$\frac{t_{fb}}{2\pi} = -A_{51}^{-1} = -\frac{\Delta_{15}}{\Delta} = -\frac{m_{out} m_a}{l_{sq} (m_J l_a + (m_a + m_J) l_{out}^*)}, \quad (\text{S13})$$

$$\alpha = -\frac{A_{15}^{-1}}{A_{11}^{-1}} = -\frac{\Delta_{51}}{\Delta_{11}} = -\frac{l(l_a + l_{out}^*) + l_a l_{out}^*}{-(l_a + l_{out}^*)} = l + \frac{l_a l_{out}^*}{(l_a + l_{out}^*)}, \quad (\text{S14})$$

$$\sigma = -\frac{A_{55}^{-1}}{A_{51}^{-1}} = -\frac{\Delta_{55}}{\Delta_{15}} = l - l_a \frac{m_J}{m_a}. \quad (\text{S15})$$

## V. SIGMA NEURON TRANSFER FUNCTION COEFFICIENTS IN THE THIRD CASE

In the third case (see Section 3.3 in the main text), the expressions for  $\Delta_{11}$  and  $\Delta_{51}$  remain the same (see Eqs. (S2)–(S3)), while one can see the appearance of terms proportional to  $m_{sq}$  in  $\Delta_{15}$  and  $\Delta_{55}$ :

$$\Delta_{15} = -m_a m_{out} + (l_a + l_{out}) m_{sq} = -m_a^* m_{out}, \quad (\text{S16})$$

$$\Delta_{55} = l[m_a m_{out} - (l_a + l_{out}) m_{sq}] - l_a[m_J m_{out} + l_{out} m_{sq}] = l[m_a^* m_{out}] - l_a[m_J^* m_{out}], \quad (\text{S17})$$

as well as in the determinant:

$$\Delta = -[l_{out} l_{sq} - m_{out}^2](m_a + m_J) - l_a l_{sq} m_J - l_a m_{out} m_{sq} = -l_{sq} (l_{out}^* m_a + (l_a + l_{out}^*) m_J) - l_a m_{out} m_{sq}. \quad (\text{S18})$$

To transform the determinant, one can substitute  $m_a = m_a^* + (l_a + l_{out}) m_{sq} / m_{out}$  and  $m_J = m_J^* - l_{out} m_{sq} / m_{out}$ :

$$\begin{aligned} \Delta &= -l_{sq} \left( l_{out}^* \left[ m_a^* + (l_a + l_{out}) \frac{m_{sq}}{m_{out}} \right] + (l_a + l_{out}^*) \left[ m_J^* - l_{out} \frac{m_{sq}}{m_{out}} \right] \right) - l_a m_{out} m_{sq} = \\ &= -l_{sq} \left( l_{out}^* m_a^* + (l_a + l_{out}^*) m_J^* \right) - l_{sq} \left( l_{out}^* (l_a + l_{out}) - (l_a + l_{out}^*) l_{out} \right) \frac{m_{sq}}{m_{out}} - l_a m_{out} m_{sq} = \\ &= -l_{sq} \left( l_{out}^* m_a^* + (l_a + l_{out}^*) m_J^* \right) - l_{sq} l_a \left( l_{out}^* - l_{out} + \frac{m_{out}^2}{l_{sq}} \right) \frac{m_{sq}}{m_{out}} = -l_{sq} \left( l_{out}^* m_a^* + (l_a + l_{out}^*) m_J^* \right). \quad (\text{S19}) \end{aligned}$$

The obtained expressions correspond to (S10)–(S11) when replacing  $m_a$  and  $m_J$  with their effective values (see Eq. (40) from the main text).

## References

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1. F. R. Gantmacher, *The theory of matrices* (AMS CHELSEA PUBLISHING, 2000).
  2. A. S. Ionin, N. S. Shuravin, L. N. Karelina, A. N. Rossolenko, M. S. Sidel'nikov, S. V. Egorov, V. I. Chichkov, M. V. Chichkov, M. V. Zhdanova, A. E. Shchegolev, and V. V. Bol'ginov, *JETP* **137**, 888 (2023).