Supplementary Material to the article "Mean pairwise distances in Rouse polymer subject to fast loop extrusion"

A. Rouse terms

As explained in the main text, in order to derive the set of equations (13) from the main text on the function $\mu(s)$ we multiply kinetic equation (9) by $\frac{1}{\gamma b^2}\Delta\boldsymbol{r}_i\Delta\boldsymbol{r}_j$, where $1 \ll i,j \ll N$ and $|j-i| = s$, and integrate it over coordinates of all beads. For the first term in the right-hand side, which corresponds to thermal diffusion of beads, we then find

$$\frac{\kappa}{\gamma b^2}\sum_{n=1}^{N}\int(\boldsymbol{r}_{i+1}-\boldsymbol{r}_i)(\boldsymbol{r}_{j+1}-\boldsymbol{r}_j)\Delta_{r_n}\mathcal{P}(\boldsymbol{r}_1,...,\boldsymbol{r}_N;t)\prod_{k=1}^{N}d^3\boldsymbol{r} = 4\delta_{ij}-2\delta_{i,j-1}-2\delta_{i,j+1}. \tag{S1}$$

Next let us consider the second term in the kinetic equation (9) which is related to harmonic interaction of the beads. If $i = j$, then

$$\frac{1}{b^2}\sum_{n=2}^{N-1}\int(\boldsymbol{r}_{i+1}-\boldsymbol{r}_i)(\boldsymbol{r}_{j+1}-\boldsymbol{r}_j)\boldsymbol{\nabla}_{r_n}[(\boldsymbol{r}_{n+1}-2\boldsymbol{r}_n+\boldsymbol{r}_{n-1})\mathcal{P}(\boldsymbol{r}_1,...,\boldsymbol{r}_N;t)]\prod_{k=1}^{N}d^3\boldsymbol{r} = \tag{S2}$$

$$= \frac{1}{b^2}\sum_{n=2}^{N-1}\int\sum_{\alpha,\beta}(x_{i+1}^{\alpha}-x_i^{\alpha})^2\partial_{x_n^{\beta}}[(x_{n+1}^{\beta}-2x_n^{\beta}+x_{n-1}^{\beta})\mathcal{P}]\prod_{k=1}^{N}d^3\vec{r} = \tag{S3}$$

$$= \frac{2}{b^2}\sum_{n=2}^{N-1}\int\sum_{\alpha,\beta}\delta_{i,n}\delta_{\alpha\beta}(x_{i+1}^{\alpha}-x_i^{\alpha})[(x_{n+1}^{\beta}-2x_n^{\beta}+x_{n-1}^{\beta})\mathcal{P}]\prod_{k=1}^{N}d^3\boldsymbol{r} - \tag{S4}$$

$$-\frac{2}{b^2}\sum_{n=2}^{N-1}\int\sum_{\alpha,\beta}\delta_{i+1,n}\delta_{\alpha\beta}(x_{i+1}^{\alpha}-x_i^{\alpha})[(x_{n+1}^{\beta}-2x_n^{\beta}+x_{n-1}^{\beta})\mathcal{P}]\prod_{k=1}^{N}d^3\boldsymbol{r} = \tag{S5}$$

$$= \frac{2}{b^2}\int\sum_{\alpha}(x_{i+1}^{\alpha}-x_i^{\alpha})[(x_{i+1}^{\alpha}-2x_i^{\alpha}+x_{i-1}^{\alpha})\mathcal{P}]\prod_{k=1}^{N}d^3\boldsymbol{r} - \frac{2}{b^2}\int\sum_{\alpha}(x_{i+1}^{\alpha}-x_i^{\alpha})[(x_{i+2}^{\alpha}-2x_{i+1}^{\alpha}+x_i^{\alpha})\mathcal{P}]\prod_{k=1}^{N}d^3\boldsymbol{r} = \tag{S6}$$

$$= \frac{2}{b^2}\int(\boldsymbol{r}_{i+1}-\boldsymbol{r}_i)[(\boldsymbol{r}_{i+1}-2\boldsymbol{r}_i+\boldsymbol{r}_{i-1})\mathcal{P}]\prod_{k=1}^{N}d^3\boldsymbol{r} - \frac{2}{b^2}\int(\boldsymbol{r}_{i+1}-\boldsymbol{r}_i)[(\boldsymbol{r}_{i+2}-2\boldsymbol{r}_{i+1}+\boldsymbol{r}_i)\mathcal{P}]\prod_{k=1}^{N}d^3\boldsymbol{r} = \tag{S7}$$

$$= \frac{2}{b^2}\int\Delta\boldsymbol{r}_i(\Delta\boldsymbol{r}_i-\Delta\boldsymbol{r}_{i-1})\mathcal{P}\prod_{k=1}^{N}d^3\boldsymbol{r} - \frac{2}{b^2}\int\Delta\boldsymbol{r}_i(\Delta\boldsymbol{r}_{i+1}-\Delta\boldsymbol{r}_i)\mathcal{P}\prod_{k=1}^{N}d^3\boldsymbol{r} = \tag{S8}$$

$$= 4\left(\mu(0)-\mu(1)\right). \tag{S9}$$

For $i = j+1$ we find

$$\frac{1}{b^2}\sum_{n=2}^{N-1}\int(\boldsymbol{r}_{i+1}-\boldsymbol{r}_i)(\boldsymbol{r}_{j+1}-\boldsymbol{r}_j)\boldsymbol{\nabla}_{r_n}[(\boldsymbol{r}_{n+1}-2\boldsymbol{r}_n+\boldsymbol{r}_{n-1})\mathcal{P}(\boldsymbol{r}_1,...,\boldsymbol{r}_N;t)]\prod_{k=1}^{N}d^3\boldsymbol{r} = \tag{S10}$$

$$= \frac{1}{b^2}\sum_{n=2}^{N-1}\int(\boldsymbol{r}_{i+1}-\boldsymbol{r}_i)(\boldsymbol{r}_i-\boldsymbol{r}_{i-1})\boldsymbol{\nabla}_{r_n}[(\boldsymbol{r}_{n+1}-2\boldsymbol{r}_n+\boldsymbol{r}_{n-1})\mathcal{P}(\boldsymbol{r}_1,...,\boldsymbol{r}_N;t)]\prod_{k=1}^{N}d^3\boldsymbol{r} = \tag{S11}$$

$$= \frac{1}{b^2}\sum_{n=2}^{N-1}\int\sum_{\alpha,\beta}(x_{i+1}^\alpha-x_i^\alpha)(x_i^\alpha-x_{i-1}^\alpha)\partial_{x_n}^\beta[(x_{n+1}^\beta-2x_n^\beta+x_{n-1}^\beta)\mathcal{P}]\prod_{k=1}^{N}d^3\boldsymbol{r} = \tag{S12}$$

$$= \frac{1}{b^2}\int\sum_{\alpha}(x_i^\alpha-x_{i-1}^\alpha)[(x_{i+1}^\alpha-2x_i^\alpha+x_{i-1}^\alpha)\mathcal{P}]\prod_{k=1}^{N}d^3\boldsymbol{r} - \tag{S13}$$

$$-\frac{1}{b^2}\int\sum_{\alpha}(x_{i+1}^\alpha-x_i^\alpha)[(x_{i+1}^\alpha-2x_i^\alpha+x_{i-1}^\alpha)\mathcal{P}]\prod_{k=1}^{N}d^3\vec{r} - \tag{S14}$$

$$-\frac{1}{b^2}\int\sum_{\alpha}(x_i^\alpha-x_{i-1}^\alpha)(x_{i+2}^\alpha-2x_{i+1}^\alpha+x_i^\alpha)\mathcal{P}\prod_{k=1}^{N}d^3\vec{r} + \tag{S15}$$

$$+\frac{1}{b^2}\int\sum_{\alpha}(x_{i+1}^\alpha-x_i^\alpha)(x_i^\alpha-2x_{i-1}^\alpha+x_{i-2}^\alpha)\mathcal{P}\prod_{k=1}^{N}d^3\vec{r} = \tag{S16}$$

$$= \frac{1}{b^2}\int(\boldsymbol{r}_i-\boldsymbol{r}_{i-1})[(\boldsymbol{r}_{i+1}-2\boldsymbol{r}_i+\boldsymbol{r}_{i-1})\mathcal{P}]\prod_{k=1}^{N}d^3\boldsymbol{r} - \tag{S17}$$

$$-\frac{1}{b^2}\int(\boldsymbol{r}_{i+1}-\boldsymbol{r}_i)[(\boldsymbol{r}_{i+1}-2\boldsymbol{r}_i+\boldsymbol{r}_{i-1})\mathcal{P}]\prod_{k=1}^{N}d^3\boldsymbol{r} - \tag{S18}$$

$$-\frac{1}{b^2}\int(\boldsymbol{r}_i-\boldsymbol{r}_{i-1})(\boldsymbol{r}_{i+2}-2\boldsymbol{r}_{i+1}+\boldsymbol{r}_i)\mathcal{P}\prod_{k=1}^{N}d^3\boldsymbol{r} + \frac{1}{b^2}\int(\boldsymbol{r}_{i+1}-\boldsymbol{r}_i)(\boldsymbol{r}_i-2\boldsymbol{r}_{i-1}+\boldsymbol{r}_{i-2})\mathcal{P}\prod_{k=1}^{N}d^3\boldsymbol{r} = \tag{S19}$$

$$= \frac{1}{b^2}\int\Delta\boldsymbol{r}_{i-1}(\Delta\boldsymbol{r}_i-\Delta\boldsymbol{r}_{i-1})\mathcal{P}\prod_{k=1}^{N}d^3\boldsymbol{r} - \frac{1}{b^2}\int\Delta\boldsymbol{r}_i(\Delta\boldsymbol{r}_i-\Delta\boldsymbol{r}_{i-1})\mathcal{P}\prod_{k=1}^{N}d^3\boldsymbol{r} - \tag{S20}$$

$$-\frac{1}{b^2}\int\Delta\boldsymbol{r}_{i-1}(\Delta\boldsymbol{r}_{i+1}-\Delta\boldsymbol{r}_i)\mathcal{P}\prod_{k=1}^{N}d^3\boldsymbol{r} + \frac{1}{b^2}\int\Delta\boldsymbol{r}_i(\Delta\boldsymbol{r}_{i-1}-\Delta\boldsymbol{r}_{i-2})\mathcal{P}\prod_{k=1}^{N}d^3\boldsymbol{r} = \tag{S21}$$

$$= -2(\mu(0)-2\mu(1)+\mu(2)). \tag{S22}$$

Finally, when $|i-j|\geq 2$, one gets

$$\frac{1}{b^2}\sum_{n=2}^{N-1}\int(\boldsymbol{r}_{i+1}-\boldsymbol{r}_i)(\boldsymbol{r}_{j+1}-\boldsymbol{r}_j)\boldsymbol{\nabla}_{r_n}[(\boldsymbol{r}_{n+1}-2\boldsymbol{r}_n+\boldsymbol{r}_{n-1})\mathcal{P}(\boldsymbol{r}_1,...,\boldsymbol{r}_N;t)]\prod_{k=1}^{N}d^3\boldsymbol{r}= \tag{S23}$$

$$=\frac{1}{b^2}\int\sum_{n=2}^{N-1}\sum_{\alpha,\beta}(x_{i+1}^{\alpha}-x_i^{\alpha})(x_{j+1}^{\alpha}-x_j^{\alpha})\partial_{x_n}^{\beta}[(x_{n+1}^{\beta}-2x_n^{\beta}+x_{n-1}^{\beta})\mathcal{P}]\prod_{k=1}^{N}d^3\boldsymbol{r}= \tag{S24}$$

$$=\frac{2}{b^2}\int\sum_{n=2}^{N-1}\sum_{\alpha,\beta}\delta_{\alpha,\beta}\delta_{i,n}(x_{j+1}^{\alpha}-x_j^{\alpha})[(x_{n+1}^{\beta}-2x_n^{\beta}+x_{n-1}^{\beta})\mathcal{P}]\prod_{k=1}^{N}d^3\boldsymbol{r}- \tag{S25}$$

$$-\frac{2}{b^2}\int\sum_{n=2}^{N-1}\sum_{\alpha,\beta}\delta_{\alpha,\beta}\delta_{i+1,n}(x_{j+1}^{\alpha}-x_j^{\alpha})[(x_{n+1}^{\beta}-2x_n^{\beta}+x_{n-1}^{\beta})\mathcal{P}]\prod_{k=1}^{N}d^3\boldsymbol{r}= \tag{S26}$$

$$=\frac{2}{b^2}\int\sum_{\alpha}(x_{j+1}^{\alpha}-x_j^{\alpha})(x_{i+1}^{\alpha}-2x_i^{\alpha}+x_{i-1}^{\alpha})\mathcal{P}\prod_{k=1}^{N}d^3\boldsymbol{r}- \tag{S27}$$

$$-\frac{2}{b^2}\int\sum_{\alpha}(x_{j+1}^{\alpha}-x_j^{\alpha})(x_{i+2}^{\alpha}-2x_{i+1}^{\alpha}+x_i^{\alpha})\mathcal{P}\prod_{k=1}^{N}d^3\boldsymbol{r}= \tag{S28}$$

$$\tag{S29}$$

$$=\frac{2}{b^2}\int(\boldsymbol{r}_{j+1}-\boldsymbol{r}_j)(\boldsymbol{r}_{i+1}-2\boldsymbol{r}_i+\boldsymbol{r}_{i-1})\mathcal{P}\prod_{k=1}^{N}d^3\boldsymbol{r}- \tag{S30}$$

$$-\frac{2}{b^2}\int(\boldsymbol{r}_{j+1}-\boldsymbol{r}_j)(\boldsymbol{r}_{i+2}-2\boldsymbol{r}_{i+1}+\boldsymbol{r}_i)\mathcal{P}\prod_{k=1}^{N}d^3\boldsymbol{r}= \tag{S31}$$

$$=\frac{2}{b^2}\int\Delta\boldsymbol{r}_j(\Delta\boldsymbol{r}_i-\Delta\boldsymbol{r}_{i-1})\mathcal{P}\prod_{k=1}^{N}d^3\boldsymbol{r}-\frac{2}{b^2}\int\Delta\boldsymbol{r}_j(\Delta\boldsymbol{r}_{i+1}-\Delta\boldsymbol{r}_i)\mathcal{P}\prod_{k=1}^{N}d^3\boldsymbol{r}= \tag{S32}$$

$$=-2\left(\mu(|j-i-1|)-2\mu(|j-i|)+\mu(|j-i+1|)\right). \tag{S33}$$

In the subsequent sections we handle the extrusion-related terms in Eq. (9) of the main text.

B. Diagram (a) (see Fig. 1 of the main text)

This diagram corresponds to configurations where the center of the looped segment and its ends are inside the interval $[i,j+1]$.

From Eq. (9) in the main text we easily find

$$\mathcal{G}_a(0)=\frac{k_{on}}{\gamma}\mu(0)\sum_{n=i}^{i+1}\sum_{l/2=0}^{\infty}\rho(l)\delta_{l,0}\approx\frac{4k_{on}}{\lambda\gamma}\mu(0), \tag{S34}$$

$$\mathcal{G}_a(1)=\frac{k_{on}}{\gamma}\mu(1)\sum_{n=i}^{i+2}\sum_{l/2=0}^{\infty}\rho(l)\delta_{l,0}\approx\frac{6k_{on}}{\lambda\gamma}\mu(1), \tag{S35}$$

$$\mathcal{G}_a(2)=\frac{k_{on}}{\gamma}\mu(2)\sum_{n=i}^{i+3}\sum_{l/2=0}^{\infty}\rho(l)\delta_{l,0}\approx\frac{8k_{on}}{\lambda\gamma}\mu(2), \tag{S36}$$

$$\mathcal{G}_a(3)\approx\frac{10k_{on}}{\lambda\gamma}\mu(3)+\frac{k_{on}}{\gamma}\rho(2)\left(\mu(1)+2\mu(2)+\mu(3)\right), \tag{S37}$$

$$\mathcal{G}_a(4)\approx\frac{12k_{on}}{\lambda\gamma}\mu(4)+2\frac{k_{on}}{\gamma}\rho(2)(\mu(3)+\mu(4)). \tag{S38}$$

Next, if $s=2m+1$, where m is integer, then for the segments of sufficiently large length $s\geq 5$ we find the following general-form expression

$$\mathcal{G}_a(s) = 2\frac{k_{on}}{\gamma}\mu(s)\sum_{n=i+2}^{i+\frac{s-1}{2}}\sum_{l/2=0}^{n-i-2}\rho(l) + \frac{k_{on}}{\gamma}\mu(s)\sum_{\frac{l}{2}=0}^{\frac{s-3}{2}}\rho(l) + 4\frac{k_{on}}{\gamma}\mu(s)\rho(0) + \frac{k_{on}}{\gamma}\rho(s-1)\sum_{k=i}^{n_c-1}\sum_{k'=n_c}^{j}\mu(k'-k) + \tag{S39}$$

$$+2\frac{k_{on}}{\gamma}\sum_{n=i+2}^{n_c-1}\rho(2(n-i-1))\sum_{k=i}^{n-1}\mu(j-k). \tag{S40}$$

The terms on the right-hand side correspond, respectively, to the loops contained within the interval $[i+2, j-1]$; to the loops of zero length corresponding to extruders seated on the beads i, $i+1$, j, and $j+1$; to the loop of length $s-1$ centred at the bead number $n_c = \frac{s-1}{2} + i + 1$ (the midpoint of the interval $[i+2, j-1]$) whose endpoints corresponds to the beads $i+1$ and j; and to the loops whose centres belong to the interval $[i+2, j-1]$ (except for the central point n_c), and one of whose endpoints corresponds to the beads $i+1$ or j.

Performing summation where it is possible we find

$$\mathcal{G}_a(s) = \frac{k_{on}}{\gamma}\mu(s)\left(s - 3 - \lambda + (\lambda+2)\left(\frac{\lambda}{\lambda+2}\right)^{\frac{s-1}{2}}\right) + \tag{S41}$$

$$+\frac{k_{on}}{\gamma}\mu(s)\left(1 - \left(\frac{\lambda}{\lambda+2}\right)^{\frac{s-1}{2}}\right) + \frac{8k_{on}}{(2+\lambda)\gamma}\mu(s) + \frac{k_{on}}{\gamma}\rho(s-1)\sum_{k=i}^{n_c-1}\sum_{k'=n_c}^{j}\mu(k'-k) + \tag{S42}$$

$$+2\frac{k_{on}}{\gamma}\sum_{n=i+2}^{n_c-1}\rho(2(n-i-1))\sum_{k=i}^{n-1}\mu(j-k) = \tag{S43}$$

$$= \frac{k_{on}}{\gamma}\left[s - 2 - \lambda + (\lambda+1)\left(\frac{\lambda}{\lambda+2}\right)^{\frac{s-1}{2}}\right]\mu(s) + \frac{8k_{on}}{(\lambda+2)\gamma}\mu(s) + \tag{S44}$$

$$+\frac{2k_{on}}{(\lambda+2)\gamma}\left(\frac{\lambda}{\lambda+2}\right)^{\frac{s-1}{2}}\left[\sum_{m=1}^{\frac{s-1}{2}} m\mu(m) + \sum_{m=\frac{s+1}{2}}^{s}(s-m+1)\mu(m)\right] + \tag{S45}$$

$$+2\frac{k_{on}}{\gamma}\left[-\left(\frac{\lambda}{\lambda+2}\right)^{\frac{s-1}{2}}\sum_{k'=\frac{s+1}{2}}^{s}\mu(k') + \frac{\lambda}{\lambda+2}(\mu(s)+\mu(s-1)) + \frac{\lambda}{\lambda+2}\sum_{m'=\frac{s+1}{2}}^{s-2}\mu(m')\left(\frac{\lambda}{\lambda+2}\right)^{s-m'-1}\right]. \tag{S46}$$

Final expression for diagram (a) when $s = 2m+1$

$$\mathcal{G}_a(s) = 2\frac{k_{on}}{\gamma}\sum_{m=0}^{s_{max}}\left\{\left(\frac{s}{2} - 1 - \frac{\lambda}{2} + \frac{\lambda+1}{2}\left(\frac{\lambda}{\lambda+2}\right)^{\frac{s-1}{2}} + \frac{4}{\lambda+2}\right)\delta_{m,s} + \right. \tag{S47}$$

$$+\frac{1}{\lambda+2}\left(\frac{\lambda}{\lambda+2}\right)^{\frac{s-1}{2}}\left(mH\left[\frac{s-1}{2} - m\right] + (s-m+1)H\left[m - \frac{s+1}{2}\right]\right) - \tag{S48}$$

$$-\left(\frac{\lambda}{\lambda+2}\right)^{\frac{s-1}{2}}H\left[m - \frac{s+1}{2}\right] + \frac{\lambda}{\lambda+2}\delta_{m,s} + \frac{\lambda}{\lambda+2}\delta_{m,s-1} + \tag{S49}$$

$$\left. +(1-\delta_{m,s-1})(1-\delta_{m,s})H\left[m - \frac{s+1}{2}\right]\left(\frac{\lambda}{\lambda+2}\right)^{s-m}\right\}H[s-k](1-\delta_{m,0})\mu(m). \tag{S50}$$

Here $H[z] = 1$ when $z \geq 0$ and $H[z] = 0$ when $z < 0$.

Let us explain how the double sums entering Eqs. (S39) and (S40) have been simplified. First of all, we passed to new summation indices $r = k$, $m = k' - k$ in the following expression

$$\sum_{k=i}^{n_c-1}\sum_{k'=n_c}^{j}\mu(k'-k)=\sum_{m=1}^{\frac{s-1}{2}}\sum_{n=n_c-m}^{n_c-1}\mu(m)+\sum_{m=\frac{s+1}{2}}^{s}\sum_{n=i}^{j-m}\mu(m)= \tag{S51}$$

$$=\sum_{m=1}^{\frac{s-1}{2}}m\mu(m)+\sum_{m=\frac{s+1}{2}}^{s}(s-m+1)\mu(m). \tag{S52}$$

Next, introducing change of indices $k'=k-i$ and $n'=n-i-1$, we obtained

$$\sum_{n=i+2}^{n_c-1}\rho(2(n-i-1))\sum_{k=i}^{n-1}\mu(j-k)=\sum_{n'=1}^{\frac{s-1}{2}-1}\rho(2n')\sum_{k'=0}^{n'}\mu(s-k'). \tag{S53}$$

To further simplify expression (S53), we applied the discrete analogue of the integration by parts method. Namely, introducing notations $a_m=q^m$, $q=\frac{\lambda}{\lambda+2}$ and $b_m=\sum_{k=0}^{m}\mu(s-k)$, we found

$$a_{m+1}b_{m+1}-a_mb_m=qa_m(b_m+\mu(s-m-1))-a_mb_m, \tag{S54}$$

and, therefore,

$$a_mb_m=\frac{a_{m+1}b_{m+1}-a_mb_m}{q-1}-\frac{q}{q-1}a_m\mu(s-m-1). \tag{S55}$$

From (S53) and (S55) one then obtains

$$\sum_{m=1}^{\frac{s-1}{2}-1}a_mb_m=\frac{1}{q-1}\sum_{m=1}^{\frac{s-1}{2}-1}(a_{m+1}b_{m+1}-a_mb_m)-\frac{q}{q-1}\sum_{m=1}^{\frac{s-1}{2}-1}a_m\mu(s-m-1)= \tag{S56}$$

$$=\frac{1}{q-1}(a_{\frac{s-1}{2}}b_{\frac{s-1}{2}}-a_1b_1)-\frac{q}{q-1}\sum_{m=1}^{\frac{s-1}{2}-1}a_m\mu(s-m-1). \tag{S57}$$

And, therefore,

$$\sum_{n'=1}^{\frac{s-1}{2}-1}\rho(2n')\sum_{k'=0}^{n'}\mu(s-k')=\frac{2}{\lambda+2}\sum_{m=1}^{\frac{s-1}{2}-1}a_mb_m= \tag{S58}$$

$$=-\left(\frac{\lambda}{\lambda+2}\right)^{\frac{s-1}{2}}\sum_{k'=\frac{s+1}{2}}^{s}\mu(k')+\frac{\lambda}{\lambda+2}(\mu(s)+\mu(s-1))+\frac{\lambda}{\lambda+2}\sum_{m'=\frac{s+1}{2}}^{s-2}\mu(m')\left(\frac{\lambda}{\lambda+2}\right)^{s-m'-1}. \tag{S59}$$

Substitution of Eqs. (S52) and (S59) into Eq. (S39)-(S40) yields Eq. (S44), (S45) и (S46).

In the complementary case $s=2m$ where m is integer, for $s\geq 6$ we find

$$\mathcal{G}_a(s)=2\frac{k_{on}}{\gamma}\mu(s)\sum_{n=i+2}^{\frac{s}{2}+i}\sum_{\frac{l}{2}=0}^{n-i-2}\rho(l)+4\frac{k_{on}}{\gamma}\mu(s)\rho(0)+2\frac{k_{on}}{\gamma}\sum_{n=i+2}^{\frac{s}{2}+i}\rho(2(n-i-1))\sum_{k=i}^{n-1}\mu(j-k). \tag{S60}$$

The given expression takes into account loops contained within the interval $[i+2,j-1]$; loops of zero length at beads i, $i+1$, j, and $j+1$; and loops whose centres belong to the interval $[i+2,j-1]$ and one of whose ends corresponds to beads $i+1$ or j. Using the standard geometric progression sum formula and the discrete counterpart of integration by parts method we find

$$\mathcal{G}_a(s)=\frac{k_{on}}{\gamma}\mu(s)\left(s-2-\lambda+(\lambda+2)(\frac{\lambda}{\lambda+2})^{s/2}\right)+\frac{8k_{on}}{(2+\lambda)\gamma}\mu(s)+2\frac{k_{on}}{\gamma}\sum_{n=i+2}^{\frac{s}{2}+i}\rho(2(n-i-1))\sum_{k=i}^{n-1}\mu(j-k)= \tag{S61}$$

$$=\frac{k_{on}}{\gamma}\mu(s)\left[s-2-\lambda+\lambda\left(\frac{\lambda}{\lambda+2}\right)^{\frac{s}{2}-1}\right]+4\frac{k_{on}}{\gamma}\mu(s)\rho(0)+ \tag{S62}$$

$$+2\frac{k_{on}}{\gamma}\frac{\lambda}{\lambda+2}(\mu(s)+\mu(s-1))-2\frac{k_{on}}{\gamma}\left(\frac{\lambda}{\lambda+2}\right)^{\frac{s}{2}}\sum_{k=0}^{s/2}\mu(s-k)+2\frac{k_{on}}{\gamma}\frac{\lambda}{\lambda+2}\sum_{m=1}^{\frac{s}{2}-1}\left(\frac{\lambda}{\lambda+2}\right)^{m}\mu(s-m-1). \tag{S63}$$

Final expression for diagram (a) when $s = 2m$

$$\mathcal{G}_a(s) = \frac{k_{on}}{\gamma} \sum_{k=0}^{s_{max}} \left[\left(s - 2 - \lambda + \lambda \left(\frac{\lambda}{\lambda+2} \right)^{\frac{s}{2}-1} + \frac{2(4+\lambda)}{2+\lambda} \right) \delta_{k,s} + \frac{2\lambda}{\lambda+2} \delta_{m,s-1} - 2 \left(\frac{\lambda}{\lambda+2} \right)^{s/2} H[m - \frac{s}{2}] + \right. \tag{S64}$$

$$\left. +2 \left(\frac{\lambda}{\lambda+2} \right)^{s-k} H[k - \frac{s}{2}](1 - \delta_{k,s-1})(1 - \delta_{k,s}) \right] H[s-k] \mu(k). \tag{S65}$$

C. Diagram (d) (see Fig. 1 of the main text)

This diagram corresponds to configurations where the center of the loop lies outside the interval $[i, j+1]$, and both ends of the loop are outside the interval $[i+1, j]$. It is convenient to divide the contribution of the diagram (d) into two parts: that due to the loops whose ends do not coincide with beads i and j, and that associated with the loops whose one end falls exactly on one of the beads i or j. Then for $s \geq 1$ we obtain

$$\mathcal{G}_d(s) = \frac{k_{on}}{\gamma} \mu(s) \sum_{n=1}^{i-1} \sum_{l/2=0}^{i-1-n} \rho(l) + \frac{k_{on}}{\gamma} \mu(s) \sum_{n=j+2}^{N} \sum_{l/2=0}^{n-j-2} \rho(l) + 2 \frac{k_{on}}{\gamma} \sum_{n=1}^{i-1} \rho(2(i-n)) \sum_{k=n}^{i} \mu(j-k). \tag{S66}$$

Performing summation of the geometric series and using indices transform ($k' = i - k$, $m = i - n$) together with discrete counterpart of integration by parts for the last term we find

$$\mathcal{G}_d(s) = \frac{k_{on}}{\gamma} \mu(s) \left(N - s - \lambda - 2 \right) + 2 \frac{k_{on}}{\gamma} \sum_{m=1}^{i-1} \rho(2m) \sum_{k'=0}^{m} \mu(s+k') \approx \tag{S67}$$

$$\approx \frac{k_{on}}{\gamma} \left(N - s - \lambda \right) \mu(s) + 2 \frac{k_{on}}{\gamma} \sum_{m=1}^{\infty} \left(\frac{\lambda}{\lambda+2} \right)^{m-1} \mu(s+m) = \tag{S68}$$

$$= \frac{k_{on}}{\gamma} N \mu(s) + \frac{k_{on}}{\gamma} \sum_{k=0}^{s_{max}} \left(-(s+\lambda) \delta_{ks} + 2 \left(\frac{\lambda}{\lambda+2} \right)^{k-s-1} H[k-s-1] \right) \mu(k). \tag{S69}$$

Let us explain the calculations involving last term in the right-hand side of Eq. (S66) in more details. Introducing notations $a_m = q^m$, $q = \frac{\lambda}{\lambda+2}$ and $b_m = \sum_{k'=0}^{m} \mu(s+k')$ we get

$$a_{m+1} b_{m+1} - a_m b_m = q a_m (b_m + \mu(s+m+1)) - a_m b_m = (q-1) a_m b_m + q a_m \mu(s+m+1), \tag{S70}$$

and, thus,

$$a_m b_m = \frac{a_{m+1} b_{m+1} - a_m b_m}{q-1} - \frac{q}{q-1} a_m \mu(s+m+1). \tag{S71}$$

From (S70) and (S71) one then finds

$$\sum_{m=1}^{i-1} a_m b_m = \frac{1}{q-1} \sum_{m=1}^{i-1} (a_{m+1} b_{m+1} - a_m b_m) - \frac{q}{q-1} \sum_{m=1}^{i-1} a_m \mu(s+m+1) = \tag{S72}$$

$$= \frac{1}{q-1} a_i b_i - \frac{1}{q-1} a_1 b_1 - \frac{q}{q-1} \sum_{m=1}^{i-1} a_m \mu(s+m+1) \approx \tag{S73}$$

$$- \frac{q}{q-1} \sum_{k=0}^{1} \mu(s+k) - \frac{q}{q-1} \sum_{m=1}^{i-1} q^m \mu(s+m+1) = \tag{S74}$$

$$= -\frac{q}{q-1} \left(\mu(s) + \mu(s+1) + \sum_{m=2}^{i} q^{m-1} \mu(s+m) \right). \tag{S75}$$

Therefore

$$\sum_{m=1}^{i-1}\rho(2m)\sum_{k'=0}^{m}\mu(s+k')=\frac{2}{\lambda+2}\sum_{m=1}^{i-1}a_m b_m=\frac{\lambda}{\lambda+2}\left(\mu(s)+\mu(s+1)+\sum_{m=2}^{i}\left(\frac{\lambda}{\lambda+2}\right)^{m-1}\mu(s+m)\right)\approx \tag{S76}$$

$$\approx\frac{\lambda}{\lambda+2}\left(\mu(s)+\mu(s+1)+\sum_{m=2}^{\infty}\left(\frac{\lambda}{\lambda+2}\right)^{m-1}\mu(s+m)\right), \tag{S77}$$

where we took into account that $i\sim N/2\gg s,\lambda$. Substitution of Eq. (S77) into (S66) gives (S69).

Let us consider separately the case $s=0$. From kinetic equation (9) in the main text we find

$$\mathcal{G}_d(0)=\frac{k_{on}}{\gamma}\mu(0)\sum_{n=1}^{i-1}\sum_{l/2=0}^{i-1-n}\rho(l)+\frac{k_{on}}{\gamma}\mu(0)\sum_{n=i+2}^{N}\sum_{l/2=0}^{n-i-2}\rho(l)+2\frac{k_{on}}{\gamma}\sum_{n=1}^{i-1}\rho(2(i-n))\sum_{k=n}^{i}\sum_{k'=n}^{i}\mu(|k'-k|). \tag{S78}$$

After summation this expression reduces to the following form

$$\mathcal{G}_d(0)=\frac{k_{on}}{\gamma}\mu(0)\left[i-1-\frac{\lambda}{2}+\frac{\lambda+2}{2}\left(\frac{\lambda}{\lambda+2}\right)^{i}\right]+\frac{k_{on}}{\gamma}\mu(0)\left[N-j-1-\frac{\lambda}{2}+\frac{\lambda+2}{2}\left(\frac{\lambda}{\lambda+2}\right)^{N-j}\right]+ \tag{S79}$$

$$+2\frac{k_{on}}{\gamma}\sum_{n=1}^{i-1}\sum_{l/2=1}^{i-n}\rho(l)\sum_{k=n}^{i}\sum_{k'=n}^{i}\mu(|k-k'|)\approx \tag{S80}$$

$$\approx\frac{k_{on}}{\gamma}\mu(0)\,(N-\lambda-2)+2\frac{k_{on}}{\gamma}\sum_{n=1}^{i-1}\rho(2(i-n))\sum_{k=n}^{i}\sum_{k'=n}^{i}\mu(|k'-k|)\approx \tag{S81}$$

$$\approx\frac{k_{on}}{\gamma}\,(N-2)\,\mu(0)+2\frac{k_{on}}{\gamma}\lambda\mu(1)+2\frac{k_{on}}{\gamma}\sum_{k=2}^{\infty}\left(\lambda+\frac{4(k-1)}{\lambda+2}\right)\left(\frac{\lambda}{\lambda+2}\right)^{k-1}\mu(k)= \tag{S82}$$

$$=\frac{k_{on}}{\gamma}N\mu(0)+\frac{k_{on}}{\gamma}\sum_{k=0}^{s_{max}}\left[-2\delta_{k,0}+2\lambda\delta_{k,1}+2\left(\lambda+\frac{4(k-1)}{\lambda+2}\right)\left(\frac{\lambda}{\lambda+2}\right)^{k-1}(1-\delta_{k,0})(1-\delta_{k,1})\right]\mu(k). \tag{S83}$$

Here we exploit that $(\frac{\lambda}{\lambda+2})^i\ll 1$ due to the condition $i\sim N/2\gg s,\lambda\geq 1$. Also we used that

$$\sum_{k=n}^{i}\sum_{k'=n}^{i}\mu(|k-k'|)=2\sum_{m=1}^{i-n}\sum_{r=n}^{i-m}\mu(m)+\sum_{r=n}^{i}\mu(m)=2\sum_{m=1}^{i-n}(i-n-m+1)\mu(m)+(i-n+1)\mu(0). \tag{S84}$$

and

$$\sum_{n=1}^{i-1}\rho(2(i-n))\sum_{k=n}^{i}\sum_{k'=n}^{i}\mu(|k-k'|)= \tag{S85}$$

$$=2\sum_{n=1}^{i-1}\rho(2(i-n))\sum_{m=1}^{i-n}(1-m)\mu(m)+2\sum_{n=1}^{i-1}\rho(2(i-n))(i-n)\sum_{m=1}^{i-n}\mu(m)+\mu(0)\sum_{n=1}^{i-1}\rho(2(i-n))(i-n+1)= \tag{S86}$$

$$=2\sum_{k=1}^{i-1}\rho(2k)\sum_{m=1}^{k}(1-m)\mu(m)+2\sum_{k=1}^{i-1}\rho(2k)k\sum_{m=1}^{k}\mu(m)+\mu(0)\sum_{k=1}^{i-1}\rho(2k)(k+1)\approx \tag{S87}$$

$$\approx\frac{\lambda}{2}\mu(0)+\lambda\mu(1)+\sum_{k=1}^{\infty}\left(\lambda+\frac{4k}{\lambda+2}\right)\left(\frac{\lambda}{\lambda+2}\right)^{k}\mu(k+1). \tag{S88}$$

To treat the double summation we exploited again the discrete analogue of integration by parts.

D. Details of numerical simulations

We finally get the following set of equations

$$\begin{cases} 4-4\left(\mu(0)-\mu(1)\right)-\frac{k_{on}}{\gamma}N\mu(0)+\mathcal{G}_a(0)+\mathcal{G}_d(0)=0, \\ -2+2(\mu(0)-2\mu(1)+\mu(2))-\frac{k_{on}}{\gamma}N\mu(1)+\mathcal{G}_a(1)+\mathcal{G}_d(1)=0, \\ 2(\mu(s-1)-2\mu(s)+\mu(s+1))-\frac{k_{on}}{\gamma}N\mu(s)+\mathcal{G}_a(s)+\mathcal{G}_d(s)=0, \quad \text{for} \quad s\geq 2. \end{cases} \tag{S89}$$

To find numerical solution it is convenient to use the matrix form of resulting problem. Namely, we can represent set of equations (S89) as

$$\hat{A}\cdot\boldsymbol{\mu}=\boldsymbol{h}, \tag{S90}$$

where $\boldsymbol{h}=(-4,2,0,...,0)^T$ and $\hat{A}$ is the square matrix of dimension $s_{max}+1$, whose elements $a_{s,k}$ are given below. Here and in what follows index s numerates rows, whereas index k corresponds to columns. Also, we introduced the notation $\chi=k_{on}/\gamma$. By $H[z]$ we denote the Heaviside function ($H[z]=1$ for $z\geq 0$, and $H[z]=0$ for $z<0$).

Matrix elements with $s=0$:

$$a_{0,k}=-4\delta_{k,0}+4\delta_{k,1}+\frac{4}{\lambda}\chi\delta_{k,0}+ \tag{S91}$$

$$+\chi\left[-2\delta_{k,0}+2\lambda\delta_{k,1}+2\left(\lambda+\frac{4(k-1)}{\lambda+2}\right)\left(\frac{\lambda}{\lambda+2}\right)^{k-1}(1-\delta_{k,0})(1-\delta_{k,1})\right]. \tag{S92}$$

Matrix elements with $s=1$:

$$a_{1,k}=2\delta_{k,0}+\left(-4+\frac{6}{\lambda}\chi-\chi(1+\lambda)\right)\delta_{k,1}+2\delta_{k,2}+2\chi\left(\frac{\lambda}{\lambda+2}\right)^{k-2}H[k-2]. \tag{S93}$$

Matrix elements with $s=2$:

$$a_{2,k}=2\delta_{k,1}+\left(-4+\frac{8}{\lambda}\chi+2\delta_{k,2}-\chi(2+\lambda)\right)\delta_{k,2}+2\chi\left(\frac{\lambda}{\lambda+2}\right)^{k-3}H[k-3]. \tag{S94}$$

Matrix elements with $s=3$:

$$a_{3,k}=\frac{2\chi\lambda}{(\lambda+2)^2}\delta_{k,1}+\left(2+\frac{4\chi\lambda}{(\lambda+2)^2}\right)\delta_{k,2}+\left(\frac{10}{\lambda}\chi+\frac{2\chi\lambda}{(\lambda+2)^2}-4-\chi(3+\lambda)\right)\delta_{k,3}+2\delta_{k,4}+ \tag{S95}$$

$$+2\chi\left(\frac{\lambda}{\lambda+2}\right)^{k-4}H[k-4]. \tag{S96}$$

Matrix elements with $s=4$:

$$a_{4,k}=\left(2+\frac{4\chi\lambda}{(\lambda+2)^2}\right)\delta_{k,3}+\left(-4+\frac{12}{\lambda}\chi+\frac{4\chi}{\lambda+2}-\chi(4+\lambda)\right)\delta_{k,4}+2\delta_{k,5}+2\chi\left(\frac{\lambda}{\lambda+2}\right)^{k-5}H[k-5]. \tag{S97}$$

Matrix elements with $s=2m+1\geq 5$:

$$a_{s,k}=2\delta_{k,s-1}-4\delta_{k,s}+2\delta_{k,s+1}+2\chi\left\{\left(\frac{s}{2}-1-\frac{\lambda}{2}+\frac{\lambda+1}{2}\left(\frac{\lambda}{\lambda+2}\right)^{\frac{s-1}{2}}+\frac{4}{\lambda+2}\right)\delta_{k,s}+\right. \tag{S98}$$

$$+\frac{1}{\lambda+2}\left(\frac{\lambda}{\lambda+2}\right)^{\frac{s-1}{2}}\left(kH\left[\frac{s-1}{2}-k\right]+(s-k+1)H\left[k-\frac{s+1}{2}\right]\right)- \tag{S99}$$

$$-\left(\frac{\lambda}{\lambda+2}\right)^{\frac{s-1}{2}}H\left[k-\frac{s+1}{2}\right]+\frac{\lambda}{\lambda+2}\delta_{k,s}+\frac{\lambda}{\lambda+2}\delta_{k,s-1}+ \tag{S100}$$

$$\left.+(1-\delta_{k,s-1})(1-\delta_{k,s})H\left[k-\frac{s+1}{2}\right]\left(\frac{\lambda}{\lambda+2}\right)^{s-k}\right\}H[s-k](1-\delta_{k,0})+ \tag{S101}$$

$$+\chi\left(-(s+\lambda)\delta_{k,s}+2\left(\frac{\lambda}{\lambda+2}\right)^{k-s-1}H[k-s-1]\right). \tag{S102}$$

Matrix elements with $s = 2m \geq 6$:

$$a_{s,k} = 2\delta_{k,s-1} - 4\delta_{k,s} + 2\delta_{k,s+1} + \chi \left[\left(s - 2 - \lambda + \lambda \left(\frac{\lambda}{\lambda+2} \right)^{\frac{s}{2}-1} + \frac{2(4+\lambda)}{2+\lambda} \right) \delta_{k,s} + \tag{S103}$$

$$+ \frac{2\lambda}{\lambda+2}\delta_{k,s-1} - 2\left(\frac{\lambda}{\lambda+2} \right)^{s/2} H\left[k - \frac{s}{2}\right] + 2\left(\frac{\lambda}{\lambda+2} \right)^{s-k} H\left[k - \frac{s}{2}\right] (1-\delta_{k,s-1})(1-\delta_{k,s}) \right] H[s-k] + \tag{S104}$$

$$+ \chi \left(-(s+\lambda)\delta_{k,s} + 2\left(\frac{\lambda}{\lambda+2} \right)^{k-s-1} H[k-s-1] \right). \tag{S105}$$