Supplementary Material to the article "On measurement of the second-order coherence of light-matter BECs with a single detector"

1 Equations for $\langle \hat{n}_{\leq K} \rangle$ and $\langle \hat{n}_{\leq K}\hat{n}_{\leq K} \rangle$ in the fast thermalization limit

In the following, we will use the theory from [1, 2]. In the fast thermalization limit, the density matrix of polaritons reduces to

$$\hat{\rho} = \sum_{N=0}^{+\infty} P_N \frac{1}{Z_N} \sum_{\sum_{\mathbf{q}} n_{\mathbf{q}}=N} e^{-\sum_{\mathbf{k}} n_{\mathbf{k}}\beta_{\mathbf{k}}} \hat{R}, \tag{S1}$$

where $\beta_{\mathbf{k}} = \hbar(\omega_{\mathbf{k}} - \omega_{\mathbf{k=0}})/k_B T$ and $\hat{R}$ is the density matrix of polaritons such that $\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}}\hat{R} = n_{\mathbf{k}}\hat{R}$. We denote the partially traced density matrix $\hat{\rho}_{\leq K} = \mathrm{tr}_{|\mathbf{k}|>K}\hat{\rho}$, where we traced all the polariton states with wave vectors exceeding K. We also denote $\hat{R}_{\leq K} = \mathrm{tr}_{|\mathbf{k}|>K}\hat{R}$ having the same meaning. From (S1), we obtain the following

$$\hat{\rho}_{\leq K} = \sum_{N=0}^{+\infty} P_N \frac{1}{Z_N} \sum_{n=0}^{N} \sum_{\sum_{|\mathbf{q}_1|\leq K} n_{\mathbf{q}_1}=n} e^{-\sum_{|\mathbf{k}_1|\leq K} n_{\mathbf{k}_1}\beta_{\mathbf{k}_1}} \hat{R}_{\leq K} \sum_{\sum_{|\mathbf{q}_2|>K} n_{\mathbf{q}_2}=N-n} e^{-\sum_{|\mathbf{k}_2|>K} n_{\mathbf{k}_2}\beta_{\mathbf{k}_2}}. \tag{S2}$$

This partially traced density matrix allows us to find the averages $\langle \hat{n}_{\leq K} \rangle = \mathrm{tr}\,(\hat{n}_{\leq K}\hat{\rho}) = \mathrm{tr}_{|\mathbf{k}|\leq K}(\hat{n}_{\leq K}\hat{\rho}_{\leq K})$ and $\langle \hat{n}_{\leq K} \rangle = \mathrm{tr}\,(\hat{n}_{\leq K}\hat{n}_{\leq K}\hat{\rho}) = \mathrm{tr}_{|\mathbf{k}|\leq K}(\hat{n}_{\leq K}\hat{n}_{\leq K}\hat{\rho}_{\leq K})$. To do this, one should use Eq. (S2) and the relation $\hat{n}_{\leq K}\hat{R}_{\leq K} = n\hat{R}_{\leq K}$. This brings us to Eq. (19) and Eq. (20), where

$$X_N = \sum_{\sum_{|\mathbf{q}|\leq K} n_{\mathbf{q}}=N} e^{-\sum_{|\mathbf{k}|\leq K} n_{\mathbf{k}}\beta_{\mathbf{k}}}, \qquad Y_N = \sum_{\sum_{|\mathbf{q}|>K} n_{\mathbf{q}}=N} e^{-\sum_{|\mathbf{k}|>K} n_{\mathbf{k}}\beta_{\mathbf{k}}}. \tag{S3}$$

We obtain the generating functions

$$\sum_{N=0}^{+\infty} X_N \xi^N = \frac{e^{G\cdot \mathrm{Ln}_2(\xi) - G\cdot \mathrm{Ln}_2(\xi e^{-\beta_K})}}{1-\xi}, \qquad \sum_{N=0}^{+\infty} Y_N \xi^N = e^{G\cdot \mathrm{Ln}_2(\xi e^{-\beta_K})}, \tag{S4}$$

where $\mathrm{Ln}_2(\xi)$ is the polylogarithm of order 2. We show a derivation of the generating function for X_N, one can do it for Y_N in the same way

$$\sum_{N=0}^{+\infty} X_N \xi^N = \sum_{N=0}^{+\infty} \xi^N \sum_{\sum_{|\mathbf{q}|\leq K} n_{\mathbf{q}}=N} e^{-\sum_{|\mathbf{k}|\leq K} n_{\mathbf{k}}\beta_{\mathbf{k}}} = \sum_{N=0}^{+\infty} \sum_{\sum_{|\mathbf{q}|\leq K} n_{\mathbf{q}}=N} e^{-\sum_{|\mathbf{k}|\leq K} n_{\mathbf{k}}(\beta_{\mathbf{k}}-\ln\xi)} = \prod_{|\mathbf{k}|\leq K} \sum_{n_{\mathbf{k}}=0}^{+\infty} e^{-n_{\mathbf{k}}(\beta_{\mathbf{k}}-\ln\xi)} =$$
$$\prod_{|\mathbf{k}|\leq K} \frac{1}{1-\xi e^{-\beta_{\mathbf{k}}}} = \exp\left(-\sum_{|\mathbf{k}|\leq K} \ln(1-\xi\beta_{\mathbf{k}})\right) = \frac{e^{G\cdot \mathrm{Ln}_2(\xi) - G\cdot \mathrm{Ln}_2(\xi e^{-\beta_K})}}{1-\xi}. \tag{S5}$$

In the derivation above we moved to the continuous limit via the density of states and used the dispersion law $\omega_{\mathbf{k}} = \omega_{\mathbf{k=0}} + \alpha\mathbf{k}^2$. The corresponding recurrent equations are

$$X_N = \frac{1}{N}\sum_{n=1}^{N}\left[1 + \frac{G}{n}\left(1 - e^{-n\beta_K}\right)\right] X_{N-n}, \qquad Y_N = \frac{1}{N}\sum_{n=1}^{N} \frac{G}{n} e^{-n\beta_K} Y_{N-n}, \tag{S6}$$

where $X_0 = 1$ and $Y_0 = 1$. We also obtain the asymptotic at $N \gg G$ and $\beta_K > 0$ $X_N \approx X_\infty \left(1 - G\cdot\ln\left(1 - e^{-\beta_K}\right) - G/N\right)$, where $X_\infty = e^{G\cdot\zeta(2) - G\cdot \mathrm{Ln}_2(e^{-\beta_K})}$, and $\zeta(2)$ is the value of zeta function at 2.

2 Equations for P_N in the fast thermalization limit

In the fast thermalization limit, when the effective thermalization rate overcomes both dissipation and pumping rates, the number of polaritons with the wave vector $\mathbf{k}$ is

$$\langle \hat{n}_{\mathbf{k}}(t) \rangle = \sum_{N=0}^{+\infty} P_N(t) \langle \hat{n}_{\mathbf{k}} \rangle_N, \tag{S7}$$

where $P_N(t)$ is the probability of finding exactly N polaritons in the system that can depend on time,

$$\langle \hat{n}_{\mathbf{k}} \rangle_0 = 0, \qquad \langle \hat{n}_{\mathbf{k}} \rangle_{N+1} = (1 + \langle \hat{n}_{\mathbf{k}} \rangle_N) e^{-\beta_{\mathbf{k}}} Z_N / Z_{N+1}, \tag{S8}$$

where $\beta_{\mathbf{k}} = \hbar(\omega_{\mathbf{k}} - \omega_{\mathbf{k}=\mathbf{0}})/k_B T$, Z_N is the partition function having the generating function $\sum_{N=0}^{+\infty} Z_N \xi^N = e^{G \cdot \mathrm{Ln}_2(\xi)}/(1-\xi)$, and $\mathrm{Ln}_2(\xi)$ is the polylogarithm of order 2. The probabilities P_N obey the equations

$$\frac{\partial P_0(t)}{\partial t} = \frac{d_0(t)}{Z_1} P_1(t) - \frac{p_0(t)}{Z_0} P_0(t), \tag{S9}$$

$$\frac{\partial P_N(t)}{\partial t} = \frac{d_N(t)}{Z_{N+1}} P_{N+1}(t) - \frac{d_{N-1}(t) + p_N(t)}{Z_N} P_N(t) + \frac{p_{N-1}(t)}{Z_{N-1}} P_{N-1}(t) \text{ for } N > 0, \tag{S10}$$

where

$$d_N = Z_{N+1} \sum_{\mathbf{k}} (\gamma_{\mathbf{k}} + \varkappa_{\mathbf{k}}(t)) \langle \hat{n}_{\mathbf{k}} \rangle_{N+1}, \qquad p_N(t) = Z_N \sum_{\mathbf{k}} \varkappa_{\mathbf{k}}(t) (1 + \langle \hat{n}_{\mathbf{k}} \rangle_N). \tag{S11}$$

We find a stationary solution of Eq. (S9)–(S10) in the case when $\varkappa_{\mathbf{k}}(t) = \varkappa_{\mathbf{k}}$ does not depend on time, $\gamma_k = \gamma$ does not depend on time. We also assume that $\varkappa_{\mathbf{k}} = \varkappa$ for $|\mathbf{k}| \le K_{\mathrm{pump}}$ and $\varkappa_{\mathbf{k}} = 0$ for $|\mathbf{k}| > K_{\mathrm{pump}}$. In this case,

$$d_N = Z_{N+1}[(\gamma + \varkappa)(N+1) - \varkappa \langle \hat{n}_{\mathrm{out}} \rangle_{N+1}], \qquad p_N = Z_N [\varkappa_{\mathrm{Pump}} + \varkappa N - \varkappa \langle \hat{n}_{\mathrm{out}} \rangle_N] \tag{S12}$$

where

$$\varkappa_{\mathrm{Pump}} = \sum_{|\mathbf{k}| \le K_{\mathrm{Pump}}} \varkappa_{\mathbf{k}}, \qquad \langle \hat{n}_{\mathrm{out}} \rangle_N = \sum_{|\mathbf{k}| > K_{\mathrm{Pump}}} \langle \hat{n}_{\mathbf{k}} \rangle_N \tag{S13}$$

are the total pumping rate over the states and the number of polaritons that are not under the incoherent pumping respectively. Above condensation threshold, one can obtain an approximate analytical expression for P_N. The stationary version of Eq. (S10) is

$$[(\gamma + \varkappa)(N+1) - \varkappa \langle \hat{n}_{\mathrm{out}} \rangle_{N+1}] P_{N+1} - [\varkappa_{\mathrm{Pump}} + (\gamma + 2\varkappa) N - 2\varkappa \langle \hat{n}_{\mathrm{out}} \rangle_N] P_N$$
$$+ [\varkappa_{\mathrm{Pump}} + \varkappa(N-1) - \varkappa \langle \hat{n}_{\mathrm{out}} \rangle_{N-1}] P_{N-1} = 0. \tag{S14}$$

Following the suggestion made in [3], we assume that the probability distribution above condensation threshold is Gaussian. The average total number of polaritons is $\langle \hat{N} \rangle = \varkappa_{\mathrm{Pump}}/\gamma$, where $\varkappa_{\mathrm{Pump}}$ is defined by Eq. (S13). Thus, σ is the only parameter we have to determine. We use Eq. (22), set $N = \langle \hat{N} \rangle$, and decompose $P_{N\pm1}$ according to $P_{N\pm1} \approx P_N \left(1 - 1/2\sigma^2\right)$. The substitution of this into Eq. (S14) allows us to find σ

$$\sigma = \sqrt{\frac{2N(\gamma + \varkappa) + \gamma - \varkappa(\langle \hat{n}_{\mathrm{out}} \rangle_{N+1} + \langle \hat{n}_{\mathrm{out}} \rangle_{N-1})}{2\gamma + 2\varkappa(2\langle \hat{n}_{\mathrm{out}} \rangle_N - \langle \hat{n}_{\mathrm{out}} \rangle_{N+1} - \langle \hat{n}_{\mathrm{out}} \rangle_{N-1})}}. \tag{S15}$$

In the numerator, we can omit γ and $\langle \hat{n}_{\mathrm{out}} \rangle_{N+1} + \langle \hat{n}_{\mathrm{out}} \rangle_{N-1}$ assuming $N \gg 1$. In the denominator, we omit $2\langle \hat{n}_{\mathrm{out}} \rangle_N - \langle \hat{n}_{\mathrm{out}} \rangle_{N+1} - \langle \hat{n}_{\mathrm{out}} \rangle_{N-1}$ because $\langle \hat{n}_{\mathrm{out}} \rangle_N$ almost do not depend on N at least above the condensation threshold as one can see in Fig.3(b). Thus, we obtain $\sigma = \sqrt{\langle \hat{N} \rangle + \langle \hat{N} \rangle^2 / G_{\mathrm{Pump}}}$.

3 Connection between $g_K^{(2)}(0)$ and $\mathcal{G}_K^{(2)}(0)$ in fast thermalization limit

In the fast thermalization limit, the light emitted by different modes is uncorrelated in the sense that $\langle \hat{a}^\dagger_{\mathbf{k}_1} \hat{a}^\dagger_{\mathbf{k}_2} \hat{a}_{\mathbf{k}_3} \hat{a}_{\mathbf{k}_4} \rangle = 0$ and $\langle \hat{a}^\dagger_{\mathbf{k}_1} \hat{a}_{\mathbf{k}_2} \rangle = 0$ when three of the wave vectors are different and $\langle \hat{a}^\dagger_{\mathbf{k}_1} \hat{a}^\dagger_{\mathbf{k}_2} \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{k}} \rangle = 0$ when $\mathbf{k}_1 \neq \mathbf{k}_2$. In what follows we use these correlations and omit the boundaries of the wave vectors implicitly assuming that $|\mathbf{k}_j| \le K$ for $j = 1, .., 4$

$$\left\langle \sum_{\mathbf{k}_1} \hat{a}^\dagger_{\mathbf{k}_1} e^{-i\varphi_{\mathbf{k}_1}(\mathbf{r})} \sum_{\mathbf{k}_2} \hat{a}_{\mathbf{k}_2} e^{i\varphi_{\mathbf{k}_2}(\mathbf{r})} \right\rangle = \sum_{\mathbf{k}} \left\langle \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} \right\rangle = \langle \hat{n}_{\le K} \rangle, \tag{S16}$$

$$\left\langle \sum_{\mathbf{k}_1} \hat{a}^\dagger_{\mathbf{k}_1} e^{-i\varphi_{\mathbf{k}_1}(\mathbf{r}_1)} \sum_{\mathbf{k}_2} \hat{a}^\dagger_{\mathbf{k}_2} e^{-i\varphi_{\mathbf{k}_2}(\mathbf{r}_2)} \sum_{\mathbf{k}_3} \hat{a}_{\mathbf{k}_3} e^{i\varphi_{\mathbf{k}_3}(\mathbf{r}_2)} \sum_{\mathbf{k}_4} \hat{a}_{\mathbf{k}_4} e^{i\varphi_{\mathbf{k}_4}(\mathbf{r}_1)} \right\rangle$$
$$= \sum_{\mathbf{k}_1 \neq \mathbf{k}_2} \left\langle \hat{a}^\dagger_{\mathbf{k}_1} \hat{a}^\dagger_{\mathbf{k}_2} \hat{a}_{\mathbf{k}_1} \hat{a}_{\mathbf{k}_2} \right\rangle e^{i[\varphi_{\mathbf{k}_1}(\mathbf{r}_2) - \varphi_{\mathbf{k}_1}(\mathbf{r}_1)]} e^{i[\varphi_{\mathbf{k}_2}(\mathbf{r}_1) - \varphi_{\mathbf{k}_2}(\mathbf{r}_2)]} + \sum_{\mathbf{k}_1 \neq \mathbf{k}_2} \left\langle \hat{a}^\dagger_{\mathbf{k}_1} \hat{a}^\dagger_{\mathbf{k}_2} \hat{a}_{\mathbf{k}_2} \hat{a}_{\mathbf{k}_1} \right\rangle + \sum_{\mathbf{k}} \left\langle \hat{a}^\dagger_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} \hat{a}_{\mathbf{k}} \right\rangle$$
$$= 2\left(\langle \hat{n}_{\le K} \hat{n}_{\le K} \rangle - \langle \hat{n}_{\le K} \rangle\right) - \left(\sum_{|\mathbf{k}| \le K} \langle \hat{n}_{\mathbf{k}} \hat{n}_{\mathbf{k}} \rangle - \langle \hat{n}_{\le K} \rangle \right), \tag{S17}$$

where we used $\varphi_{\mathbf{k}}(\mathbf{r}_1)-\varphi_{\mathbf{k}}(\mathbf{r}_2)=\pi/2$ and relations

$$\sum_{\mathbf{k}_1\neq\mathbf{k}_2}\left\langle\hat{a}^\dagger_{\mathbf{k}_1}\hat{a}^\dagger_{\mathbf{k}_2}\hat{a}_{\mathbf{k}_1}\hat{a}_{\mathbf{k}_2}\right\rangle=\sum_{\mathbf{k}_1,\mathbf{k}_2}\left\langle\hat{a}^\dagger_{\mathbf{k}_1}\hat{a}^\dagger_{\mathbf{k}_2}\hat{a}_{\mathbf{k}_1}\hat{a}_{\mathbf{k}_2}\right\rangle-\sum_{\mathbf{k}}\left\langle\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{k}}\right\rangle=$$
$$\sum_{\mathbf{k}_1,\mathbf{k}_2}\left\langle\hat{a}^\dagger_{\mathbf{k}_1}\hat{a}_{\mathbf{k}_1}\hat{a}^\dagger_{\mathbf{k}_2}\hat{a}_{\mathbf{k}_2}\right\rangle-\sum_{\mathbf{k}}\left\langle\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}\right\rangle-\sum_{\mathbf{k}}\left\langle\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{k}}\right\rangle=\langle\hat{n}_{\leq K}\hat{n}_{\leq K}\rangle-\langle\hat{n}_{\leq K}\rangle-\sum_{\mathbf{k}}\left\langle\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{k}}\right\rangle, \quad \text{(S18)}$$

$$\sum_{\mathbf{k}_1\neq\mathbf{k}_2}\left\langle\hat{a}^\dagger_{\mathbf{k}_1}\hat{a}^\dagger_{\mathbf{k}_2}\hat{a}_{\mathbf{k}_2}\hat{a}_{\mathbf{k}_1}\right\rangle=\sum_{\mathbf{k}_1,\mathbf{k}_2}\left\langle\hat{a}^\dagger_{\mathbf{k}_1}\hat{a}^\dagger_{\mathbf{k}_2}\hat{a}_{\mathbf{k}_2}\hat{a}_{\mathbf{k}_1}\right\rangle-\sum_{\mathbf{k}}\left\langle\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{k}}\right\rangle=$$
$$\sum_{\mathbf{k}_1,\mathbf{k}_2}\left\langle\hat{a}^\dagger_{\mathbf{k}_1}\hat{a}_{\mathbf{k}_1}\hat{a}^\dagger_{\mathbf{k}_2}\hat{a}_{\mathbf{k}_2}\right\rangle-\sum_{\mathbf{k}}\left\langle\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}\right\rangle-\sum_{\mathbf{k}}\left\langle\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{k}}\right\rangle=\langle\hat{n}_{\leq K}\hat{n}_{\leq K}\rangle-\langle\hat{n}_{\leq K}\rangle-\sum_{\mathbf{k}}\left\langle\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{k}}\right\rangle, \quad \text{(S19)}$$

$$\sum_{\mathbf{k}}\left\langle\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{k}}\right\rangle=\sum_{\mathbf{k}}\langle\hat{n}_{\mathbf{k}}\hat{n}_{\mathbf{k}}\rangle-\langle\hat{n}_{\leq K}\rangle. \quad \text{(S20)}$$

We substitute (S16)–(S17) into Eq.(7) using Eq.(5) and obtain

$$g^{(2)}_K(0)=2\frac{\langle\hat{n}_{\leq K}\hat{n}_{\leq K}\rangle-\langle\hat{n}_{\leq K}\rangle}{\langle\hat{n}_{\leq K}\rangle^2}-\frac{\sum_{|\mathbf{k}|\leq\mathbf{K}}\langle\hat{n}_{\mathbf{k}}\hat{n}_{\mathbf{k}}\rangle-\langle\hat{n}_{\leq K}\rangle}{\langle\hat{n}_{\leq K}\rangle^2} \quad \text{(S21)}$$

which is equivalent to Eq. (21).

4 Asymptotic for $g^{(2)}_{\mathbf{k}=\mathbf{0}}(0)$ above the condensation threshold

Above the condensation threshold, $\langle\hat{N}\rangle\gg G\ln\langle\hat{N}\rangle$,

$$\langle\hat{n}_{\mathbf{k}=\mathbf{0}}\hat{n}_{\mathbf{k}=\mathbf{0}}\rangle_N-\langle\hat{n}_{\mathbf{k}=\mathbf{0}}\rangle_N\approx\langle\hat{n}_{\mathbf{k}=\mathbf{0}}\rangle_N^2(1+\frac{G}{N})\approx N^2-2GN\ln N+GN,\qquad\langle\hat{n}_{\mathbf{k}=\mathbf{0}}\rangle_N\approx N-G\ln N. \quad \text{(S22)}$$

Therefore,

$$\langle\hat{n}_{\mathbf{k}=\mathbf{0}}\hat{n}_{\mathbf{k}=\mathbf{0}}\rangle-\langle\hat{n}_{\mathbf{k}=\mathbf{0}}\rangle=\sum_{N=0}^{+\infty}P_N\left(\langle\hat{n}_{\mathbf{k}=\mathbf{0}}\hat{n}_{\mathbf{k}=\mathbf{0}}\rangle_N-\langle\hat{n}_{\mathbf{k}=\mathbf{0}}\rangle_N\right)\approx\sum_{N=0}^{+\infty}P_N(N^2-2GN\ln N+GN)$$
$$\approx\langle\hat{N}\rangle^2+\sigma^2-2G\langle\hat{N}\rangle\ln\langle\hat{N}\rangle+G\langle\hat{N}\rangle\approx\langle\hat{N}\rangle^2+\frac{\langle\hat{N}\rangle^2}{G_{\text{Pump}}}-2G\langle\hat{N}\rangle\ln\langle\hat{N}\rangle+G\langle\hat{N}\rangle, \quad \text{(S23)}$$

$$\langle\hat{n}_{\mathbf{k}=\mathbf{0}}\rangle^2\approx\left(\sum_{N=0}^{+\infty}P_N\langle\hat{n}_{\mathbf{k}=\mathbf{0}}\rangle_N\right)^2\approx\langle\hat{N}\rangle^2-2G\langle\hat{N}\rangle\ln\langle\hat{N}\rangle. \quad \text{(S24)}$$

The substitution of these two equation into the definition of $g^{(2)}_{\mathbf{k}=\mathbf{0}}(0)$ that follows from Eq. (8)

$$g^{(2)}_{\mathbf{k}=\mathbf{0}}(0)=\frac{\langle\hat{n}_{\mathbf{k}=\mathbf{0}}\hat{n}_{\mathbf{k}=\mathbf{0}}\rangle-\langle\hat{n}_{\mathbf{k}=\mathbf{0}}\rangle}{\langle\hat{n}_{\mathbf{k}=\mathbf{0}}\rangle^2} \quad \text{(S25)}$$

leads to Eq. (23).

5 Asymptotic for $g^{(2)}_\infty(0)$ above the condensation threshold

To derive the asymptotics for $g^{(2)}_\infty(0)$, i.e. $g^{(2)}_K(0)$ at $K\to\infty$, we use Eq. (21). This equation contains three different correlations: $\langle\hat{n}_{\leq K}\rangle$, $\langle\hat{n}_{\leq K}\hat{n}_{\leq K}\rangle$, and $\sum_{|\mathbf{k}|\leq K}\langle\hat{n}_{\mathbf{k}}\hat{n}_{\mathbf{k}}\rangle$. We analyze each of this correlations in turn. We start with the correlation $\langle\hat{n}_{\leq K}\rangle$, for which we have at $K\to\infty$

$$\langle\hat{n}_{\leq K}\rangle=\sum_{N=0}^{+\infty}P_N\sum_{|\mathbf{k}|\leq K}\langle\hat{n}_{\mathbf{k}}\rangle_N\approx\sum_{N=0}^{+\infty}P_NN=\langle\hat{N}\rangle \quad \text{(S26)}$$

where we used the approximation $\sum_{|\mathbf{k}|\leq K}\langle\hat{n}_{\mathbf{k}}\rangle_N \approx N$ and Eq. (22). Similarly, we consider the correlation at $K\to\infty$

$$\langle\hat{n}_{\leq K}\hat{n}_{\leq K}\rangle = \sum_{N=0}^{+\infty} P_N\langle\hat{n}_{\leq K}\hat{n}_{\leq K}\rangle_N \approx \sum_{N=0}^{+\infty} P_N N^2 = \langle\hat{N}\rangle^2+\sigma^2 = \langle\hat{N}\rangle^2+\langle\hat{N}\rangle+\frac{\langle\hat{N}\rangle^2}{G_{\text{Pump}}} \tag{S27}$$

where we used the approximation $\langle\hat{n}_{\leq K}\hat{n}_{\leq K}\rangle_N \approx N^2$ and Eq. (22). Finally, we consider

$$\sum_{|\mathbf{k}|\leq K}\langle\hat{n}_{\mathbf{k}}\hat{n}_{\mathbf{k}}\rangle = \sum_{N=0}^{+\infty} P_N \sum_{|\mathbf{k}|\leq K}\langle\hat{n}_{\mathbf{k}}\hat{n}_{\mathbf{k}}\rangle_N. \tag{S28}$$

We obtain

$$\langle\hat{n}_{\mathbf{k}}\hat{n}_{\mathbf{k}}\rangle_N = \frac{1}{Z_N}\sum_{n=0}^{N-1}[2(N-n)-1]Z_n e^{-(N-n)\beta_{\mathbf{k}}}, \tag{S29}$$

where $\beta_{\mathbf{k}} = (\hbar\omega_{\mathbf{k}}-\hbar\omega_{\mathbf{k}=\mathbf{0}})/k_BT$. From Eq. (S29), we can prove that

$$\sum_{|\mathbf{k}|\leq K}\langle\hat{n}_{\mathbf{k}}\hat{n}_{\mathbf{k}}\rangle_N = \langle\hat{n}_{\mathbf{k}=\mathbf{0}}\hat{n}_{\mathbf{k}=\mathbf{0}}\rangle_N + \langle\hat{n}_{\mathbf{k}=\mathbf{0}}\rangle_N - \langle\hat{n}_{\leq K}\rangle_N + 2G\left(\langle\hat{n}_{\mathbf{k}=\mathbf{0}}\rangle_N - \langle\hat{n}_{|\mathbf{k}|=K}\rangle_N\right), \tag{S30}$$

where $\langle\hat{n}_{|\mathbf{k}|=K}\rangle_N$ is $\langle\hat{n}_{\mathbf{k}}\rangle_N$ at $|\mathbf{k}|=K$. To find the asymptotic for $\sum_{|\mathbf{k}|\leq K}\langle\hat{n}_{\mathbf{k}}\hat{n}_{\mathbf{k}}\rangle$, we use Eq. (S30) and the following approximate expressions that are valid above the condensation threshold

$$\langle\hat{n}_{\mathbf{k}=\mathbf{0}}\hat{n}_{\mathbf{k}=\mathbf{0}}\rangle \approx \sum_{N=0}^{+\infty} P_N(N^2-2GN\ln N+NG) \approx \langle\hat{N}\rangle^2+\sigma^2-2GN\ln N \approx \langle\hat{N}\rangle^2-2G\langle\hat{N}\rangle\ln\langle\hat{N}\rangle+\frac{\langle\hat{N}\rangle^2}{G_{\text{Pump}}}, \tag{S31}$$

$$\langle\hat{n}_{\mathbf{k}=\mathbf{0}}\rangle \approx \sum_{N=0}^{+\infty} P_N N = \langle\hat{N}\rangle, \qquad \langle\hat{n}_{\leq K}\rangle \approx \sum_{N=0}^{+\infty} P_N N = \langle\hat{N}\rangle, \qquad \langle\hat{n}_{|\mathbf{k}|=K}\rangle \approx \sum_{N=0}^{+\infty} P_N\langle\hat{n}_{|\mathbf{k}|=K}\rangle_N \approx \frac{1}{e^{\beta_{|\mathbf{k}|=K}}-1}, \tag{S32}$$

where we used Eq. (S22). As a result, we obtain

$$\sum_{|\mathbf{k}|\leq K}\langle\hat{n}_{\mathbf{k}}\hat{n}_{\mathbf{k}}\rangle \approx \langle\hat{N}\rangle^2-2G\langle\hat{N}\rangle\ln\langle\hat{N}\rangle+\frac{\langle\hat{N}\rangle^2}{G_{\text{Pump}}}. \tag{S33}$$

Finally, we substitute Eq. (S26), Eq. (S27), and Eq. (S33) into Eq. (21) and obtain Eq. (24).

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