Supplementary Material to the article "On thermal Hall effect in boson systems"

In this part we provide a detailed derivation of our result for the quantity $M_1^{\mu\nu}$ in (20). We note that our index n in Eq. (15) corresponds to both the wave-vector in the Brillouin zone and the spin wave band number n in notation of [1]. The identification of our Υ_n with the matrix $T_{\mathbf{k}}$ shows the equivalence of the first line in our (20) with Eq. (D3) in [1].

We consider the quantity

$$M_1^{\mu\nu} = -\tfrac{1}{2}\langle \hat{\mathcal{H}}\sigma_3 r^\nu \hat{V}^\mu + \hat{V}^\mu r^\nu \sigma_3 \hat{\mathcal{H}}\rangle_0 \,, \tag{S1}$$

whose average may be written as

$$\begin{aligned} M_1^{\mu\nu} &= -\frac{1}{2}\sum_n \mathrm{Tr}\left[\begin{pmatrix} N(\varepsilon_n), & 0 \\ 0, & N(\varepsilon_n)+1 \end{pmatrix} \cdot (\hat{\mathcal{H}}\sigma_3 r^\nu \hat{V}^\mu + \hat{V}^\mu r^\nu \sigma_3 \hat{\mathcal{H}})_{nn}\right] \\ &= -\frac{1}{2}\sum_n \mathrm{Tr}\left[\varepsilon_n \begin{pmatrix} N(\varepsilon_n), & 0 \\ 0, & N(-\varepsilon_n) \end{pmatrix} \cdot (r^\nu \hat{V}^\mu + \hat{V}^\mu r^\nu)_{nn}\right] . \end{aligned} \tag{S2}$$

The 2×2 matrix $(\ldots)_{nn}$ here may be represented as follows

$$\begin{aligned} \mathcal{C}_{nn} &\equiv \left(\hat{\mathcal{H}}\sigma_3 r^\nu \hat{V}^\mu + \hat{V}^\mu r^\nu \sigma_3 \hat{\mathcal{H}}\right)_{nn} = \int d\mathbf{r}\ \Upsilon_n^\dagger \left(\hat{\mathcal{H}}\sigma_3 r^\nu \hat{V}^\mu + \hat{V}^\mu r^\nu \sigma_3 \hat{\mathcal{H}}\right)\Upsilon_n \\ &= \varepsilon_n \sum_m \left[\sigma_3 \int d\mathbf{r}\, \left(\Upsilon_n^\dagger r^\nu \sigma_3 \Upsilon_m\right)\sigma_3 \int d\mathbf{r}' \left(\Upsilon_m^\dagger \hat{V}^\mu \Upsilon_n\right) + \int d\mathbf{r}' \left(\Upsilon_n^\dagger \hat{V}^\mu \Upsilon_m\right)\sigma_3 \int d\mathbf{r}\,\left(\Upsilon_m^\dagger \sigma_3 r^\nu \Upsilon_n\right)\sigma_3\right] \\ &= \varepsilon_n \sum_m \left[\sigma_3 \left(\sigma_3 r^\nu\right)_{nm} \sigma_3 (\hat{V}^\mu)_{mn} + (\hat{V}^\mu)_{nm}\sigma_3 \left(\sigma_3 r^\nu\right)_{mn}\sigma_3\right] \end{aligned} \tag{S3}$$

Now using Eq. (11) we express $(\sigma_3 r^\nu)_{nm}$ in terms of the matrix elements of $\hat{V}^\nu$

$$\begin{aligned} \sigma_3(\sigma_3 r^\nu)_{nm}\sigma_3 &= \frac{-i}{\varepsilon_n^2 - \varepsilon_m^2}\left[\varepsilon_n (\hat{V}^\nu)_{nm}\sigma_3 + \sigma_3 (\hat{V}^\nu)_{nm}\varepsilon_m\right] , \\ \sigma_3(\sigma_3 r^\nu)_{mn}\sigma_3 &= \frac{i}{\varepsilon_n^2 - \varepsilon_m^2}\left[\varepsilon_n\, \sigma_3 (\hat{V}^\nu)_{mn} + \varepsilon_m\, (\hat{V}^\nu)_{mn}\sigma_3\right] , \end{aligned} \tag{S4}$$

so that Eq. (S3) becomes

$$\begin{aligned} \mathcal{C}_{nn} &= \sum_m \frac{i\,\varepsilon_n}{\varepsilon_n^2 - \varepsilon_m^2}\left[-\big(\varepsilon_n (\hat{V}^\nu)_{nm}\sigma_3 + \sigma_3 (\hat{V}^\nu)_{nm}\varepsilon_m\big)(\hat{V}^\mu)_{mn} + (\hat{V}^\mu)_{nm}\big(\varepsilon_n\,\sigma_3(\hat{V}^\nu)_{mn} + \varepsilon_m\,(\hat{V}^\nu)_{mn}\sigma_3\big)\right] \\ &= \sum_m \frac{i\,\varepsilon_n}{\varepsilon_n^2 - \varepsilon_m^2}\left[\varepsilon_n\big((\hat{V}^\mu)_{nm}\sigma_3(\hat{V}^\nu)_{mn} - (\hat{V}^\nu)_{nm}\sigma_3(\hat{V}^\mu)_{mn}\big) + \varepsilon_m\big(\sigma_3(\hat{V}^\mu)_{nm}(\hat{V}^\nu)_{mn} - (\hat{V}^\nu)_{nm}(\hat{V}^\mu)_{mn}\sigma_3\big)\right] . \end{aligned} \tag{S5}$$

Using a formula similar to (24), we represent $\begin{pmatrix} N(\varepsilon_n), & 0 \\ 0, & N(\varepsilon_n)+1 \end{pmatrix} = (N(\varepsilon_n) + \tfrac{1}{2})\sigma_0 - \tfrac{1}{2}\sigma_3$. The structure of matrices $(\hat{V}^\mu)_{nm}$, Eq. (13), results in $\mathrm{Tr}[\mathcal{C}_{nn}\sigma_3] = 0$. We obtain at this stage

$$M_1^{\mu\nu} = -\tfrac{1}{2}\sum_n \left(N(\varepsilon_n) + \tfrac{1}{2}\right)\mathrm{Tr}\left(\mathcal{C}_{nn}\right) \tag{S6}$$

Simple calculation shows that

$$\begin{aligned} i\,\mathrm{Tr}\left((\hat{V}^\mu)_{nm}\sigma_3(\hat{V}^\nu)_{mn} - (\hat{V}^\nu)_{nm}\sigma_3(\hat{V}^\mu)_{mn}\right) &= 2\left[X_{a,nm}^{\mu\nu} + X_{b,nm}^{\mu\nu}\right] , \\ i\,\mathrm{Tr}\left(\sigma_3(\hat{V}^\mu)_{nm}(\hat{V}^\nu)_{mn} - (\hat{V}^\nu)_{nm}(\hat{V}^\mu)_{mn}\sigma_3\right) &= 2\left[X_{a,nm}^{\mu\nu} - X_{b,nm}^{\mu\nu}\right] , \end{aligned} \tag{S7}$$

with combinations $X_{c,nm}^{\mu\nu}$ introduced in (16). Finally, we obtain

$$M_1^{\mu\nu} = -\sum_{n,m}\left(N(\varepsilon_n) + \frac{1}{2}\right)\varepsilon_n\left[\frac{X_{a,nm}^{\mu\nu}}{\varepsilon_n - \varepsilon_m} + \frac{X_{b,nm}^{\mu\nu}}{\varepsilon_n + \varepsilon_m}\right] , \tag{S8}$$

which is Eq. (20) in the main text. It looks different from Eq. (D13) in [1].

[1] R. Matsumoto, R. Shindou, and S. Murakami, Thermal Hall effect of magnons in magnets with dipolar interaction, Phys. Rev. B **89**, 054420 (2014).